Journal of Engineering Science and Technology
Vol. 6, No. 4 (2011) 459 - 468

TWO-DIMENSIONAL ANALYTICAL SOLUTIONS FOR POINT SOURCE CONTAMINANTS TRANSPORT IN SEMI-INFINITE HOMOGENEOUS POROUS MEDIUM

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Abstract

An analytical solution is obtained for the advection-dispersion of a constant input concentration along unsteady horizontal flow in semi-infinite shallow aquifer. Exponential form of time-dependent seepage velocity and first order decay are considered. Point source pollutant reaches the groundwater level from a surface and moves vertically down-wards, from its each point and starts spreading in the horizontal plane. Two separates transformations are introduced. Laplace technique is used to obtain the analytical solution of the present problem.

Keywords: Advection, Dispersion, Groundwater, Shallow aquifer, Seepage velocity.

1. Introduction

The intensive size of natural resources and the large production of wastes in modern society often pose a threat to the groundwater quality and already have resultant in many incidents of groundwater contamination. Degradation of groundwater quality can take place over large areas from plane or diffuse sources like deep percolation from intensively farmed fields, or it can be caused by point sources such as septic tank, garbage disposal sites, cemeteries, mine spoils and oil spoils or other accidental entry of pollutants into the underground environment. Another possibility is contamination by line sources of poor quality water, like seepage from polluted streams or intrusion of salt water, from oceans [1, 2].

Analytical solutions in one-dimensional problems through semi-infinite or finite porous media have been presented by several researchers. A non-exhaustive list of references must include at least the works of Bastian and Lapidus [3], Banks and Ali [4], Ogata [5], Marino [6] and van Genucheten [7]. Most of these papers reveal idealistic assumptions, such as porous medium of constant porosity, seepage flow and dispersion. In deviating from the above ideal conditions, Shamir and Harleman [2] presented an analytical solution in two layered porous medium. Banks and Jerasate [8], Rumer [9] and Yadav et al. [10] considered dispersion along unsteady flow. Al-Niami and Rushton [1] considered uniform flow where as Kumar [11] took unsteady flow, against the dispersion in finite porous media. Most of these works have included the attenuation effect due to adsorption, first order radio-active decay and/or chemical reactions.

Bruce and Street [12] considered both longitudinal and lateral dispersion in semi-infinite non-adsorbing porous media in steady flow field for a constant input concentration. Shen [13] presented a generalized closed form solution for three-dimensional dispersion in saturated ambient porous media resulting from sources of finite extent with time-dependent input concentration. Hunt [14] has given one-, two- and three-dimensional solution for instantaneous, continuous and steady pollution sources in uniform groundwater. Al-Niami and Rushton [15] obtained the analytical solutions considering longitudinal and lateral dispersion in a two-layered porous medium. Güven et al. [16] used the Aris moment method to analysis the dispersion of consecutive solute in a horizontal stratified aquifer. Prakash [17] presented analytical solutions to predict temporal and spatial distribution of concentration in one-, two-, and three-dimensionally fully saturated uniform porous media flow for a point, line or parallelopiped source in an isotropic porous medium. Latinopoulos et al. [18] studied the chemical transport in two-dimensional aquifer. Ellsworth and Butters [19] discussed three-dimensional solutions used for transport problems involving arbitrary Cartesian coordinate systems.

Logan and Zlotnik [20] obtained solutions of the convection-diffusion equation with decay for periodic boundary conditions on a semi-infinite domain. Aral and Liao [21] examined solutions to two-dimensional advection-dispersion equations with time- dependent dispersion coefficients. In particular, they developed instantaneous and continuous point source solutions for constant, linear, asymptotic, and exponentially varying dispersion coefficients. Wortmanna et al. [22] presented an analytical solution for advection-diffusion equation to simulate the pollutant dispersion in the planetary boundary layer. Sirin [23] assumed pore flow velocity to be a non-divergence free, unsteady and non-stationary random function of space and time for groundwater contaminant transport in a heterogeneous medium. The significance of the new velocity correction term is investigated on a two dimensional transport problem driven by a density dependent flow. Smedt [24] presented a model for solute transport in rivers including transient storage in hyporheic zones. The model consists of an advection-dispersion equation for transport in the main channel with a sink term describing diffusive solute transfer to the hyporheic zone. The system of equations is solved analytically for instantaneous injection of a conservative tracer in an infinite uniform river reach with steady flow [21].

But all such two or three-dimensional dispersion studies have been done along steady and unidirectional or longitudinal (perpendicular to the vertical) flow field. In almost all the solutions derived in two-dimension so for only longitudinal component of velocity where considered, neglecting vertical (transverse) component, while in present study both the dispersion components along longitudinal and lateral directions and velocity along these two directions are also considered. The seepage velocity is exponentially decreasing function of time. Porous medium is considered homogeneous, isotropic saturated and of semi-infinite in horizontal plane. The seepage velocity is exponentially decreasing function of time. The first order decay term which is proportional to velocity is also considered. Analytical solution is obtained for uniform input source concentration with the help of Laplace transformation technique.

2. Mathematical Formulation of the Problem

Let the pollutant invades the groundwater level from point source. The pollutant being of a significantly higher density than the groundwater moves towards the bottom of the shallow aquifer along vertically downward, from its each point the pollutant is bound to spread in the horizontal plane along the unsteady groundwater flow. Let at one such point is the source concentration. The longitudinal and lateral (transverse) directions in the horizontal plane extend up to infinity where no concentration at any time. Let u [LT^{-1}] and v [LT^{-1}] be component of horizontal and lateral flow velocity and D_x [L^2T^{-1}] and D_y [L^2T^{-1}] be the dispersion coefficients along longitudinal and lateral directions respectively. The general partial differential equation describing hydrodynamic dispersion in homogenous, isotropic porous media can be written as

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} - u(t)\frac{\partial c}{\partial x} - v(t)\frac{\partial c}{\partial y} - \gamma(t)c \tag{1}$$

where γ [T^{-1}] is the first order decay. Let

$$u = u_o e^{-mt}, \; v = v_o e^{-mt}, \text{ and } \gamma = \gamma_o e^{-mt} \tag{2}$$

where u_o and v_o are initial velocity components along x and y axes respectively, $c(x, y, t)$ is the concentration at any time t in horizontal plane and γ_o is the first order decay constant. Rumer [9] established a relationship for steady and unsteady flow with an exponentially or sinusoidal varying flow velocity. So let

$$D_x = \alpha u \text{ and } D_y = \alpha v \tag{3}$$

where α is the coefficient having the dimension [L] and depends upon pore geometry and average pore size diameter of the porous medium. These two can be written by using of Eq. (2) as

$$D_x = D_{x_o} e^{-mt} \text{ and } D_y = D_{y_o} e^{-mt} \tag{4}$$

where $D_{x_o} = \alpha u_o$ and $D_{y_o} = \alpha v_o$ are initial dispersion coefficient components along the two respective directions. Initial and boundary conditions for present problem are as follows,

$$c = 0,\ t = 0,\ x \ge 0 \text{ , and } y \ge 0 \tag{5}$$

i.e., initially the groundwater is solute free,

$$c = c_o,\ t > 0,\ x = 0 \text{ , and } y = 0 \tag{6}$$

and $\frac{\partial c}{\partial x}=0$, $\frac{\partial c}{\partial y}=0$, $t \geq 0, x \to \infty$ and $y \to \infty$ (7)

The condition (6) indicates that the concentration is continuous across the inlet boundary and (7) indicates that there is no solute flux at end of both boundaries. Using Eqs. (2) and (4) the differential equation (1) can be written as,

$$e^{mt}\frac{\partial c}{\partial t}=D_{x_o}\frac{\partial^2 c}{\partial x^2}+D_{y_o}\frac{\partial^2 c}{\partial y^2}-u_o\frac{\partial c}{\partial x}-v_o\frac{\partial c}{\partial y}-\gamma_o c \tag{8}$$

Introducing the new time variable T by following transformation (Crank [25])

$$T=\int_0^t v(t)dt, \quad \text{where } v(t)=e^{mt} \tag{9}$$

or $T=\int_0^t e^{-mt}dt=\frac{1}{m}\left(1-e^{-mt}\right)$

For an expression e^{-mt} which is taken such that $e^{-mt}=1$ for $m = 0$ or $t = 0$, the new time variable obtained from Eq. (9) satisfies the conditions $T = 0$ for $t = 0$ and $T = \infty$ for $m = 0$. The first condition ensures that the nature of the initial condition does not change in the new time variable domain.

Thus Eq. (8) can be written as,

$$\frac{\partial c}{\partial T}=D_{x_o}\frac{\partial^2 c}{\partial x^2}+D_{y_o}\frac{\partial^2 c}{\partial y^2}-u_o\frac{\partial c}{\partial x}-v_o\frac{\partial c}{\partial y}-\gamma_o c \tag{10}$$

Let a new space variable is introduced as follows:

$$X=x+y\sqrt{\frac{D_{y_o}}{D_{x_o}}} \tag{11}$$

Therefore differential equation (10) reduces into

$$\frac{\partial c}{\partial T}=D\frac{\partial^2 c}{\partial X^2}-U\frac{\partial c}{\partial X}-\gamma_o c \tag{12}$$

where $D=D_{x_o}\left(1+\frac{D_{y_o}^2}{D_{x_o}^2}\right)$ and $U=u_o\left(u_o+v_o\sqrt{\frac{D_{y_o}}{D_{x_o}}}\right)$

After taking care of the transformations (9) and (11) the initial and boundary conditions (5)-(7) can be put as follows

$c = 0, T = 0, X \geq 0$ (13)

$c = c_o, T > 0, X = 0$ (14)

$\frac{\partial c}{\partial X}=0$, $T \geq 0, X \to \infty$ (15)

Introducing a new dependent variable $K(X, T)$ by following transformation,

$$c(X,T)=K(X,T)\exp\left\{\frac{U}{2D}X-\left(\frac{U^2}{4D}+\gamma_o\right)T\right\} \quad (16)$$

and applying Laplace transformation on Eqs. (12), (14) and (15) and using initial condition (13), we can get following ordinary boundary value problem,

$$\frac{d^2\overline{K}}{dX^2}=\frac{p\overline{K}}{D} \quad (17)$$

$$\overline{K}=\frac{c_o}{p-\frac{U^2}{4D}},\ X\to\infty \quad (18)$$

and $\frac{dg}{dx}=0,\ X\to\infty$ (19)

where $\overline{K}(X,p)=\int_0^\infty K(X,T)e^{-pT}dT$ and p is the Laplace parameter.

The solution of Eq. (17) by using the conditions (18) and (19), becomes

$$\overline{K}(X,p)=\frac{c_o}{p-\frac{U^2}{4D}}e^{-X\sqrt{p/D}} \quad (20)$$

Applying inverse Laplace transformation on (20) and then using Eq. (16), the final solution of the present problem may be written as,

$$c(X,T)=\frac{1}{2}\left[\exp\left\{\frac{\left(\beta-\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X-\sqrt{U^2+4\gamma_o DT}}{2\sqrt{DT}}\right\}\right]$$
$$+\frac{1}{2}\left[\exp\left\{\frac{\left(\beta+\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X+\sqrt{U^2+4\gamma_o DT}}{2\sqrt{DT}}\right\}\right] \quad (21)$$

where $\beta^2=\frac{U^2}{4D}$, $D=D_{x_o}\left(1+\frac{D_{y_o}^2}{D_{x_o}^2}\right)$, $U=u_o+v_o\sqrt{\frac{D_{y_o}}{D_{x_o}}}$, $X=x+y\sqrt{\frac{D_{y_o}}{D_{x_o}}}$ and $T=\frac{1}{m}\left(1-e^{-mt}\right)$.

3. Particular Cases

(i) When $m = 0$ then $T = t$ in the solution (21), the obtained solution along steady horizontal flow may be written as,

$$c(X,t)=\frac{1}{2}\left[\exp\left\{\frac{\left(\beta-\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X-\sqrt{U^2+4\gamma_o Dt}}{2\sqrt{Dt}}\right\}\right]$$

$$+\frac{1}{2}\left[\exp\left\{\frac{\left(\beta+\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X+\sqrt{U^2+4\gamma_o Dt}}{2\sqrt{Dt}}\right\}\right] \tag{22}$$

where all the variables defined above are same.

(ii) The solution for one-dimensional dispersion along unsteady flow through semi-infinite porous media can be obtained by substituting $v_o = 0$ and $D_{y_o} = 0$ in Eq. (21), i.e.

$$c(X,T)=\frac{1}{2}\left[\exp\left\{\frac{\left(\beta-\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D_{x_o}}}\right\}erfc\left\{\frac{X-\sqrt{u_o^2+4\gamma_o D_{x_o}T}}{2\sqrt{D_{x_o}T}}\right\}\right]$$

$$+\frac{1}{2}\left[\exp\left\{\frac{\left(\beta+\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D_{x_o}}}\right\}erfc\left\{\frac{X+\sqrt{u_o^2+4\gamma_o D_{x_o}T}}{2\sqrt{D_{x_o}T}}\right\}\right] \tag{23}$$

where $\beta^2=\frac{u_o^2}{4D_{x_o}}$, $X=x$ and u_o, D_{x_o}, and T represent initial velocity, dispersion coefficient along longitudinal direction (x-axis) and time variable respectively.

(iii) Concentration distribution due to longitudinal and lateral dispersion but along longitudinal unsteady porous medium flow (as considered by the previous workers) can be obtained by substituting $v_o = 0$ in the solution (21), i.e.,

$$c(X,T)=\frac{1}{2}\left[\exp\left\{\frac{\left(\beta-\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X-\sqrt{u_o^2+4\gamma_o DT}}{2\sqrt{DT}}\right\}\right]$$

$$+\frac{1}{2}\left[\exp\left\{\frac{\left(\beta+\sqrt{\beta^2+\gamma_o}\right)X}{\sqrt{D}}\right\}erfc\left\{\frac{X+\sqrt{u_o^2+4\gamma_o D_{x_o}T}}{2\sqrt{DT}}\right\}\right] \tag{24}$$

where $\beta^2=\frac{u_o^2}{4D}$, D and T are same as above.

4. Numerical Examples and Discussion

To illustrate the concentration distribution of the obtained analytical solution (21) in two-dimensional homogeneous porous medium in semi-infinite domain, an example has been chosen in which the different variables are assigned numerical values, where longitudinal and lateral seepage velocity and dispersion coefficients are $u_o = 0.95$ (m/day), $v_o = 0.095$ (m/day), $D_{x_o} = 1.05$ (m^2/day) and $D_{y_o} = 0.105$ (m^2/day) respectively. The flow resistance coefficient $m = 0.1$ (1/day) and first order decay term $\gamma_o = 0.04$ (1/day) have been chosen. Figure 1 is drawn for solution (21) with different time $t = 1.5$, 2.0 and 2.5 (days) for unsteady and Fig. 2 for steady flow ($m = 0$) respectively.

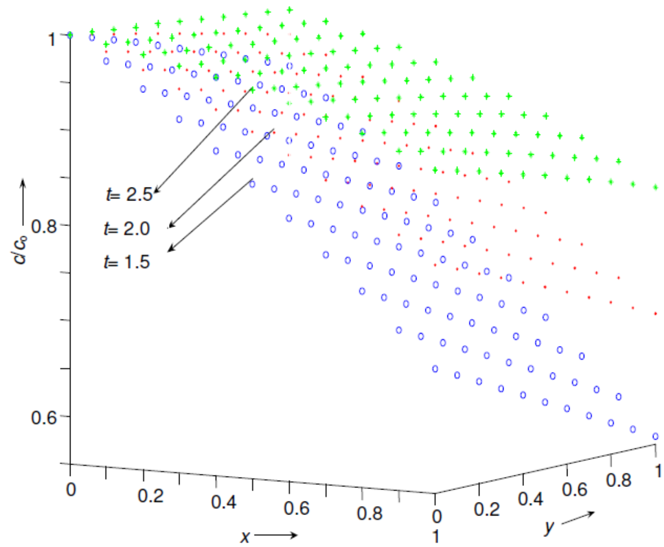

Fig. 1. Concentration Profile for Decreasing Function at Different Time for Solution (21).

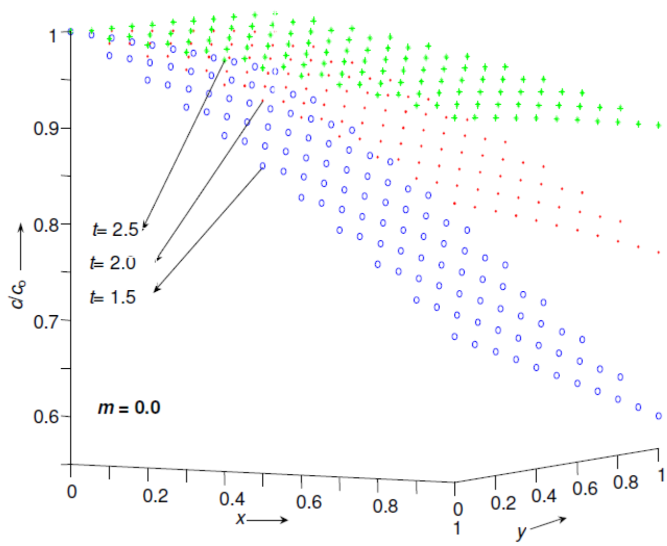

Fig. 2. Concentration Profile for Constant Coefficients at Different Time for Solution (22).

Figure 3 shows comparison between unsteady and steady flow at time $t = 2.0$ (days). The behavior of the concentrations in both directions increases with increasing time, whereas the concentration values at particular position in unsteady flow are lower than the steady flow.

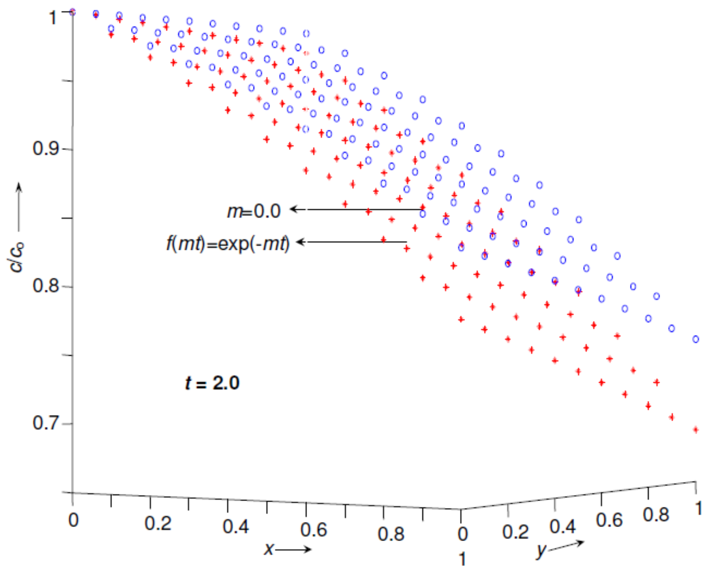

Fig. 3. Comparison of Concentration Profiles between Decreasing Function and Constant Coefficients for Solution (21) and (22), $t = 2.0$ (days).

The lateral velocity component is considered one-tenth of the longitudinal component. Although, it significant is small but can not be ignore for better accuracy of result. This effect decreases with time. It means as the time increases dispersion process goes on dominating over the transport due to convection. In the case of unsteady flow similar but lesser effect would appear too. Though exponential form of seepage velocity is considered in the present analysis, there is no reason why other form of seepage velocity could not be used as long as the boundary conditions $c(x, y, t)$ and $c(x, y, T)$ are compatible. A horizontal two-dimensional transport model is selected because the groundwater flow is essentially horizontal and vertically mixing of the groundwater contaminant is a good approximation in such shallow aquifer. The time dependent behavior of pollutants in subsurface is of interest for many realistic problems where the concentration is observed or needs to be predicted at fixed positions. Problems of solute transport in two-dimension involving sequential first order decay reactions frequently occurs in soil, chemical engineering and groundwater systems, for example the migration of simultaneous movement of interacting nitrogen species, organic phosphate transport and the transport of pesticides and their metabolites. The accuracy of the numerical method is validated by direct comparisons with the analytical results.

Conclusions

This study presents an analytical solution to solve the advection–dispersion equation with longitudinal and transverse dispersion for describing the two-dimensional solute transport in a homogeneous porous media. In the derived solution, both the components (longitudinal and transverse) of velocity are assumed exponentially deceasing function of time, dispersion coefficient and first order decay are directly proportional to velocity. The hypothetical studies indicate that the effect of pollutant is not uniform but decrease as we move away from origin along either direction or horizontal plane. The governing solute transport equation is solved analytically by employing Laplace Transformation Technique (LTT).

The derived solution is an effective and useful for further application to verify the newly developed numerical transport model for predicting the two-dimensional time-dependent transport of contaminants. The application results reveal that the solute transport process at the test site obeys the linearly time-dependent dispersion model and that the linearly time-dependent assumption is valid in this real world example. The proposed solution can be applied to field problems where the hydrological properties of the medium and prevailing boundary and initial conditions are the same as, or can be approximated by, the ones considered in this study.

Acknowledgements

The author thanks the two reviewers, whose helpful comments and suggestions greatly improved the paper. The second author gratefully acknowledges the financial assistance in the form of UGC- Dr. D. S. Kothari Post Doctoral Fellowship.

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