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INCLUSION PROPERTIES OF WEIGHTED WEAK ORLICZ SPACES

AL AZHARY MASTA[1,*], IFRONIKA[2],
MUHAMMAD TAQIYUDDIN[3], RIZKY ROSJANUARDI[1]

[1]Departemen of Mathematics Education, Universitas Pendidikan Indonesia,
Jl. Dr. Setiabudhi No. 229 Bandung, Indonesia
[2]Analysis Geometry Group, Faculty of Mathematics and Natural Sciences,
Bandung Institute of Technology, Jl. Ganesha 10, Bandung, Indonesia
[3]Department of Mathematics and Science Education, University of Georgia,
Athens, Georgia 30602, United States
[*]Corresponding author: alazhari.masta@upi.edu

Abstract

Weak Lebesgue spaces and weighted weak Lebesgue spaces are valuable for limiting the boundedness of some integral operators on Lebesgue spaces and weighted Lebesgue spaces. This paper discusses multiple structures of weighted weak Lebesgue spaces and weighted weak Orlicz spaces on $\mathbb{R}^n$. Our findings include necessary and sufficient conditions for inclusions between weighted weak Lebesgue spaces. In addition, we acquired similar results in the case of weighted weak Orlicz spaces. One of the key methods to obtain our findings is to leverage the norm of the characteristic functions of the balls in $\mathbb{R}^n$. Our result can be seen as a counterpart of the results on inclusion properties of weak Morrey spaces, Orlicz spaces, and weak Orlicz spaces.

Keywords: Inclusion property, Weighted weak Lebesgue spaces, Weighted weak Orlicz spaces.

1. Introduction

Orlicz spaces are a generalization of Lebesgue spaces, which Z. W. Birnbaum and W. Orlicz firstly introduced in 1931 [1, 2]. The research on both Lebesgue spaces and Orlicz spaces has been investigated by numerous researchers in the last few decades [3-6]. One of the important discussions within the existing literature is about inclusions among those spaces. In 1989, Maligranda and Mastyło [4] scrutinized the inclusion properties pertained in Orlicz spaces. In 2016, Masta et al. [7] acquired necessary and sufficient conditions for inclusion relations in Orlicz spaces and weak Orlicz spaces by utilizing a different technique employed by Maligranda. More recent results regarding inclusions can be found in [8-10], which discuss the third version of weak Orlicz-Morrey spaces in 2019 [11], discrete Oricz-Morrey spaces in 2021 [9], and generalized Orlicz spaces in 2020 [10].

Of our interest is that in [4], it can be seen that two Orlicz spaces and two weak Orlicz spaces are comparable with reference to Young functions for any measurable set. However, the Lebesgue spaces L_p cannot be compared with respect to the number p. Because comparing two Lebesgue spaces L_p cannot be achieved for different p values, Osançliol [12] examined sufficient and necessary conditions for inclusion relations on weighted Orlicz spaces to obtain a comparison between two different weighted Lebesgue spaces. However, the results on sufficient and necessary conditions for weak Lebesgue spaces and weighted weak Orlicz space have not been established by Osançliol. Thus, based on that research gap and all the above rationales, we are interested in studying a unification of weak Lebesgue spaces, weighted weak Lebesgue spaces, and weak Orlicz spaces, namely weighted weak Orlicz spaces. The novelties of this study are our findings that include necessary and sufficient conditions for inclusions on weighted weak Orlicz spaces and Hölder's inequality on those spaces. As a direct consequence of our result, the inclusions on weighted weak Lebesgue spaces and weak Orlicz space follow as a special case.

First, we will review the notion of Orlicz spaces and weak Orlicz spaces, starting with the definition. Let $\Psi: [0,\infty) \to [0,\infty)$ be a Young function [that is, Ψ is convex, $\lim_{t\to 0}\Psi(t) = 0 = \Psi(0)$, left-continuous and $\lim_{t\to\infty}\Psi(t) = \infty$], *the Orlicz space* $L_\Psi(\mathbb{R}^n)$ is the set of measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \Psi(c|f(x)|)dx < \infty$$

for some $c > 0$. The space $L_\Psi(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$\| f \|_{L_\Psi(\mathbb{R}^n)} := \inf\left\{b > 0: \int_{\mathbb{R}^n} \Psi\left(\frac{|f(x)|}{b}\right) dx \le 1\right\}.$$

Note that, $L_\Psi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ for $\Psi(t) := t^p$, $1 \le p < \infty$. Meanwhile, for a Young function Ψ, *the weak Orlicz space* $wL_\Psi(\mathbb{R}^n)$ is the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{wL_\Psi(\mathbb{R}^n)} := \inf\left\{b > 0: \sup_{t>0}\Psi(t) \left|\left\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\right\}\right| \le 1\right\} < \infty.$$

Furthermore, if $\Psi(t) := t^p$ for some $p \ge 1$ then $wL_\Psi(\mathbb{R}^n) = wL_p(\mathbb{R}^n)$ is weak Lebesgue space.

Let us now move to discussing the weighted Orlicz spaces as well as the weighted weak Orlicz spaces. We will start by reviewing the definitions of the two spaces. For a Young function Ψ and a weight u on $\mathbb{R}^n$ (i.e., $u: \mathbb{R}^n \to (0, \infty)$ is measurable function), *the weighted Orlicz space* $L_\Psi^u(\mathbb{R}^n)$ is the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that $uf \in L_\Psi(\mathbb{R}^n)$. Note that the space $L_\Psi^u(\mathbb{R}^n)$ is Banach spaces equipped with the norm

$$\| f \|_{L_\Psi^u(\mathbb{R}^n)} := \| uf \|_{L_\Psi(\mathbb{R}^n)} = \inf\left\{ b > 0: \int_{\mathbb{R}^n} \Psi\left(\frac{|u(x)f(x)|}{b}\right) dx \leq 1 \right\}.$$

On the other hand, to discuss the later space, we need a Young function Ψ and a weight u on $\mathbb{R}^n$. Then, our definition of *the weighted weak Orlicz space* $wL_\Psi^u(\mathbb{R}^n)$ is the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that $uf \in wL_\Psi(\mathbb{R}^n)$. If, otherwise, we define

$$\| f \|_{wL_\Psi^u(\mathbb{R}^n)} := \inf\left\{ b > 0: \sup_{t>0} \Psi(t) |\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{b} > t\}| \leq 1 \right\} = \| uf \|_{wL_\Psi(\mathbb{R}^n)},$$

then $\|\cdot\|_{wL_\Psi^u(\mathbb{R}^n)}$ is quasi-norm on $wL_\Psi^u(\mathbb{R}^n)$. This quasi-norm is known as a weighted weak Orlicz quasi-norm. Accordingly, the space $wL_\Psi^u(\mathbb{R}^n)$ is quasi-Banach spaces equipped with the quasi-norm $\|\cdot\|_{wL_\Psi^u(\mathbb{R}^n)}$. If $u(x) = 1$ for every $x \in \mathbb{R}^n$, then $wL_\Psi^u(\mathbb{R}^n) = wL_\Psi(\mathbb{R}^n)$ is weak Orlicz space. Meanwhile, for $\Psi(t) := t^p$ $(1 \leq p < \infty)$, the space $wL_\Psi^u(\mathbb{R}^n) = wL_p^u(\mathbb{R}^n)$ is the weighted weak Lebesgue space . We define *the weighted weak Lebesgue space* $wL_p^u(\mathbb{R}^n)$ as the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\| f \|_{wL_p^u(\mathbb{R}^n)} := \sup_{t>0} t |\{x \in \mathbb{R}^n: |u(x)f(x)| > t\}|^{\frac{1}{p}} < \infty.$$

Let $u_1, u_2: \mathbb{R}^n \to (0, \infty)$, we denote $u_1 \preceq u_2$ if there exists a constant $B > 0$ such that $u_1(x) \leq B u_2(x)$ for all $x \in \mathbb{R}^n$. Note that, if $u_1 \preceq u_2$ then $\| f \|_{L_\Psi^{u_1}(\mathbb{R}^n)} \leq B \| f \|_{L_\Psi^{u_2}(\mathbb{R}^n)}$ and $\| f \|_{wL_\Psi^{u_1}(\mathbb{R}^n)} \leq B \| f \|_{wL_\Psi^{u_2}(\mathbb{R}^n)}$.

Regarding the relationship between weighted weak Orlicz spaces and (strong) weighted Orlicz spaces, the following lemma 1 gives us a clear insight.

Lemma 1.1 *Let Ψ be a Young function and $u: \mathbb{R}^n \to (0, \infty)$ be a measurable function. Then $L_\Psi^u(\mathbb{R}^n) \subset wL_\Psi^u(\mathbb{R}^n)$ with $\| f \|_{wL_\Psi^u(\mathbb{R}^n)} \leq \| f \|_{L_\Psi^u(\mathbb{R}^n)}$ for every $f \in L_\Psi^u(\mathbb{R}^n)$.*

Meanwhile, Osançliol proved the relationship between two (strong) weighted Orlicz spaces as elaborated in the following theorem 1 [8].

Theorem 1.2 *Let Ψ be continuous Young function satisfying the $\triangle_2$ condition [that is, there exists $K > 0$ such that $\Psi(2t) \leq K\Psi(t)$ for all $t \geq 0$], and u_1, u_2 are measurable functions such that $u_i(x + y) \leq u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1,2$. Then the following statements are equivalent:*

(1) $u_1 \preceq u_2$.

(2) $L_\Psi^{u_2}(\mathbb{R}^n) \subseteq L_\Psi^{u_1}(\mathbb{R}^n)$.

(3) There exists $B > 0$ such that $\| f \|_{L_\Psi^{u_1}(\mathbb{R}^n)} \leq B \| f \|_{L_\Psi^{u_2}(\mathbb{R}^n)}$, for every $f \in L_\Psi^{u_2}(\mathbb{R}^n)$.

In connection with and motivated by Theorem 1, we shall investigate the inclusion relation between weighted weak Orlicz spaces by proving its necessary and sufficient conditions and compare it with the result for weighted Orlicz spaces.

Before presenting our result, we will discuss two lemmas, which are necessary for proving our main results, in the following section.

2. Methods

In this section, we discuss our methods in obtaining our results by reviewing some mathematical lemmas used in the previous research. Notably, to achieve our purpose, we use similar methods employed in [7, 13-15], which pay close attention and consideration to the characteristic functions of open balls in $\mathbb{R}^n$. In doing so, we recall some lemmas which are needed to prove our primary results later in the next section.

Lemma 1.3 *Let Ψ be a Young function and $\Psi^{-1}(s) := inf\{r \geq 0 : \Psi(r) > s\}$. Some basic* properties of Ψ^{-1} are [11]
(1) $\Psi^{-1}(s) = 0$ if and only if $s = 0$.
(2) Ψ^{-1} is a nondicreasing function.
(3) $\Psi(\Psi^{-1}(s)) \leq s$ and $\Psi^{-1}(\Psi(s)) \geq s$ for nonnegative s.

Lemma 1.4 *Let Ψ_1, Ψ_2 be Young functions. For any $s > 0$, if there exists* [15] *$B_1, B_2 > 0$ such that $\Psi_2^{-1}(s) \leq B_1 \Psi_1^{-1}(B_2 s)$, then we have $\Psi_1(\frac{t}{B_1}) \leq B_2 \Psi_2(t)$ for $t = \Psi_2^{-1}(s)$.*

Lemma 1.4 *Let $u : \mathbb{R}^n \to (0, \infty)$ be a measurable function such that $u(x + y) \leq u(x) \cdot u(y)$ for every $x, y \in \mathbb{R}^n$ and $1 \leq p < \infty$. If $L_x f$ is translation function, i.e $L_x f(y) := f(y - x)$ for every $y \in \mathbb{R}^n$, then:*
(1) For all $f \in wL_p^u(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$, we have $\| L_x f \|_{wL_p^u(\mathbb{R}^n)} \leq u(x) \| f \|_{wL_p^u(\mathbb{R}^n)}$.
(2) If $f \in wL_p^u(\mathbb{R}^n)$ and $f \neq 0$, then there exists a constant $B > 0$ (depends on f) such that

$$\frac{u(x)}{B} \leq \| L_x f \|_{wL_p^u(\mathbb{R}^n)} \leq Bu(x).$$

Let us discuss how we will use constants in this paper. We use the letter B to indicate constants that may differ from line to line. Meanwhile, constants with subscripts, such as B_1,B_2, stay the same in different lines.

3. Results and Discussions

In this section, we prove some necessary and sufficient conditions for the inclusion relation of weighted weak Lebesgue spaces. The main idea for proving sufficiency is the quasi-monotonicity of the parameter. The necessary condition of the inclusion is obtained by calculating the quasi-norm of the characteristic function on the ball. In general, we will present the following five primary results:

1. Necessary and sufficient conditions for inclusion properties of weighted weak Lebesgue spaces
2. Sufficient and necessary conditions for inclusions between the spaces $wL_{\Psi_1}^u(\mathbb{R}^n)$ and $wL_{\Psi_2}^u(\mathbb{R}^n)$ with respect to Young functions Ψ_1, Ψ_2
3. Necessary and sufficient conditions for inclusion relation between the spaces $wL_{\Psi_1}^{u_1}(\mathbb{R}^n)$ and $wL_{\Psi_2}^{u_2}(\mathbb{R}^n)$ with respect to Young functions Ψ_1, Ψ_2 and weights u_1, u_2

4. Sufficient conditions for Hölder's inequality in weighted weak Orlicz spaces
5. Inclusion properties of weighted weak Lebesgue spaces $wL^{u_1}_{p_1}(X)$ and $wL^{u_2}_{p_2}(X)$ with respect to distinct values of p_1 and p_2 as well as u_1 and u_2.

First, our main result is dealing with the sufficient and necessary conditions for inclusion properties of weighted weak Lebesgue spaces. The following theorem will present the corresponding result.

Theorem 2.1 *Let* $1 \leq p < \infty$ *and* $u_1, u_2: \mathbb{R}^n \to (0, \infty)$ *are measurable functions such that* $u_i(x+y) \leq u_i(x) \cdot u_i(y)$ *for every* $x, y \in \mathbb{R}^n$, *where* $i = 1,2$. *Then the following statements are equivalent:*
(1) $u_1 \preceq u_2$.
(2) $wL^{u_2}_p(\mathbb{R}^n) \subseteq wL^{u_1}_p(\mathbb{R}^n)$.
(3) There exists $B > 0$ such that

$$\| f \|_{wL^{u_1}_p(\mathbb{R}^n)} \leq B \| f \|_{wL^{u_2}_p(\mathbb{R}^n)},$$

for every $f \in wL^{u_2}_p(\mathbb{R}^n)$.
Proof.

Assume that (1) holds, then there exists a constant $B > 0$ such that $u_1(x) \leq Bu_2(x)$ for every $x \in \mathbb{R}^n$. Let f be an element of $wL^{u_2}_p(\mathbb{R}^n)$. Now, take an arbitrary $t > 0$, observe that (by setting $t_1 = \frac{t}{B}$)

$$\begin{aligned} t|\{x \in \mathbb{R}^n: |u_1(x)f(x)| > t\}|^{\frac{1}{p}} &\leq t|\{x \in \mathbb{R}^n: |Bu_2(x)f(x)| > t\}|^{\frac{1}{p}} \\ &= t|\{x \in \mathbb{R}^n: |u_2(x)f(x)| > \tfrac{t}{B}\}|^{\frac{1}{p}} \\ &= Bt_1|\{x \in \mathbb{R}^n: |u_2(x)f(x)| > t_1\}|^{\frac{1}{p}} \\ &\leq B \| f \|_{wL^{u_2}_p(\mathbb{R}^n)}. \end{aligned}$$

Since $t > 0$ is arbitrary, we have $\| f \|_{wL^{u_1}_p(\mathbb{R}^n)} \leq B \| f \|_{wL^{u_2}_p(\mathbb{R}^n)}$. Next, because $wL^{u_1}_p(\mathbb{R}^n)$ and $wL^{u_2}_p(\mathbb{R}^n)$ are quasi-Banach spaces, as mentioned in [11, Appendix G], [3, Lemma 3.3] still holds for quasi-Banach spaces. Thus, we obtain that (2) and (3) are equivalent. So, it remains to show that (3) implies (1). Assume that (3) holds. Considering Lemma 1.5, we can get

$$\frac{u_1(x)}{B} \leq \| L_x f \|_{wL^{u_1}_p(\mathbb{R}^n)} \leq B \| L_x f \|_{wL^{u_2}_p(\mathbb{R}^n)} \leq Bu_2(x),$$

for every $x \in \mathbb{R}^n$. So, we can conclude that $u_1 \preceq u_2$.

Note that the relation $u_1 \preceq u_2$ is a necessary and sufficient condition for the inclusion properties of weighted weak Lebesgue spaces.

In regards to the second main result, we will investigate the inclusion characteristics of weighted weak Orlicz spaces. For getting a result, we attend to estimating the norm of the characteristic function of an open ball in $\mathbb{R}^n$ as in the following lemma.

Lemma 2.2 [5, 8] *Let* Ψ *be a Young function,* $a \in \mathbb{R}^n$, *and* $r > 0$ *be arbitrary. Then we have* $\left\|\frac{\chi_{B_a(r)}}{u}\right\|_{wL^u_\Psi(\mathbb{R}^n)} = \| \chi_{B_a(r)} \|_{wL_\Psi(\mathbb{R}^n)} = \frac{1}{\Psi^{-1}(\frac{1}{|B_a(r)|})}$ *where* $|B_a(r)|$ *denotes the volume of open ball* $B_a(r)$ *centered at* $a \in \mathbb{R}^n$ *with radius* $r > 0$.

Now we are ready to examine the inclusions between $wL^u_{\Psi_1}(\mathbb{R}^n)$ and $wL^u_{\Psi_2}(\mathbb{R}^n)$ with respect to Young functions Ψ_1, Ψ_2. Given two Young functions Ψ_1, Ψ_2, we write $\Psi_1 \prec \Psi_2$ if there exists a constant $B > 0$ such that $\Psi_1(t) \leq \Psi_2(Ct)$ for all $t > 0$.

Theorem 2.3 *Let* Ψ_1, Ψ_2 *be Young functions and* $u: \mathbb{R}^n \to (0, \infty)$ *be measurable function. Then the following statements are equivalent:*
(1) $\Psi_1 \prec \Psi_2$.
(2) $wL^u_{\Psi_2}(\mathbb{R}^n) \subseteq wL^u_{\Psi_1}(\mathbb{R}^n)$.
(3) There exists a constant $B > 0$ such that $\| f \|_{wL^u_{\Psi_1}(\mathbb{R}^n)} \leq B \| f \|_{wL^u_{\Psi_2}(\mathbb{R}^n)}$, for every $f \in wL^u_{\Psi_2}(\mathbb{R}^n)$.

Proof. Assume that (1) holds. Let $f \in wL^u_{\Psi_2}(\mathbb{R}^n)$, define

$$A_{\Psi_1,u} := \{b > 0: \sup_{t>0} \Psi_1(t) \left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{b} > t\right\}\right| \leq 1\}$$

and

$$A_{\Psi_2,u} := \left\{b > 0: \sup_{t>0} \Psi_2(Bt) \left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{b} > t\right\}\right| \leq 1\right\}$$
$$= \left\{b > 0: \sup_{s>0} \Psi_2(s) \left|\left\{x \in \mathbb{R}^n: \frac{B|u(x)f(x)|}{b} > s\right\}\right| \leq 1\right\}.$$

Observe that, for arbitrary $b \in A_{\Psi_2,u}$ and $t > 0$, we have

$$\begin{aligned}\Psi_1(t)|\{x \in \mathbb{R}^n: \tfrac{|u(x)f(x)|}{b} > t\}| &\leq \Psi_2(Bt)|\{x \in \mathbb{R}^n: \tfrac{|u(x)f(x)|}{b} > t\}| \\ &\leq \sup_{s>0} \Psi_2(s)|\{x \in \mathbb{R}^n: \tfrac{B|u(x)f(x)|}{b} > s\}| \\ &\leq 1.\end{aligned}$$

Since $t > 0$ is arbitrary, we obtain

$$\| f \|_{wL^u_{\Psi_1}(\mathbb{R}^n)} := \inf A_{\Psi_1,u} \leq \inf A_{\Psi_2,u} := \| Bf \|_{wL^u_{\Psi_2}(\mathbb{R}^n)} = B \| f \|_{wL^u_{\Psi_2}(\mathbb{R}^n)},$$

which also proves that $wL^u_{\Psi_2}(\mathbb{R}^n) \subseteq wL^u_{\Psi_1}(\mathbb{R}^n)$. Next, using a similar argument in the proof of Theorem 2.1, we have that (2) and (3) are equivalent.

Assume now that (3) holds. By Lemma 2.2, we have

$$\frac{1}{\Psi_1^{-1}\left(\frac{1}{|B_a(r)|}\right)} = \left\|\frac{\chi_{B_a(r)}}{u}\right\|_{wL^u_{\Psi_1}(\mathbb{R}^n)} \leq B \left\|\frac{\chi_{B_a(r)}}{u}\right\|_{wL^u_{\Psi_2}(\mathbb{R}^n)} = \frac{B}{\Psi_2^{-1}\left(\frac{1}{|B_a(r)|}\right)}$$

or $B\Psi_1^{-1}(\frac{1}{|B_a(r)|}) \geq \Psi_2^{-1}(\frac{1}{|B_a(r)|})$, for arbitrary $a \in \mathbb{R}^n$ and $r > 0$. By Lemma 1.4, we have

$$\Psi_1(\frac{t_0}{B}) \leq \Psi_2(t_0),$$

for $t_0 = \Psi_2^{-1}(\frac{1}{|B_a(r)|})$. Since $a \in \mathbb{R}^n$ and $r > 0$ are arbitrary, we conclude that $\Psi_1(t) \leq \Psi_2(Bt)$ for every $t > 0$.

For $u(x) = 1$, Theorem 2.3 reduces to Theorem 3.3 in [7].

For the third primary result, we scrutinize the sufficient and necessary conditions for inclusion relation between weighted weak Orlicz spaces $wL^{u_1}_{\Psi_1}(\mathbb{R}^n)$

and $wL^{u_2}_{\Psi_2}(\mathbb{R}^n)$ in regards to two Young functions Ψ_1, Ψ_2 and two weights u_1, u_2. To get the result, we need the following lemmas.

Lemma 2.4 *Let Ψ be a Young function. If $f \in wL^u_\Psi(\mathbb{R}^n)$, then for arbitrary $\epsilon > 0$ we have*

$$\sup_{t>0}\Psi(t)\left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon} > t\right\}\right| \leq 1.$$

Proof.

Suppose $f \in wL^u_\Psi(\mathbb{R}^n)$. Take an arbitrary $\epsilon > 0$, then there exists $b_\epsilon > 0$ such that $b_\epsilon \leq \| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon$ and

$$\sup_{t>0}\Psi(t)\left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{b_\epsilon} > t\right\}\right| \leq 1.$$

Because $\frac{|u(x)f(x)|}{b_\epsilon} \geq \frac{|u(x)f(x)|}{\|f\|_{wL^u_\Psi(\mathbb{R}^n)}+\epsilon}$, we have

$$\Psi(t)\left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon} > t\right\}\right| \leq \Psi(t)\left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{b_\epsilon} > t\right\}\right| \leq 1$$

for every $t > 0$. Since $t > 0$ is arbitrary, we can conclude that

$$\sup_{t>0}\Psi(t)\left|\left\{x \in \mathbb{R}^n: \frac{|u(x)f(x)|}{\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon} > t\right\}\right| \leq 1$$

for every $\epsilon > 0$.

Lemma 2.5 *Let $u: \mathbb{R}^n \to (0, \infty)$ be a measurable function such that $u(x+y) \leq u(x) \cdot u(y)$ for every $x, y \in \mathbb{R}^n$. If Ψ is a Young function, then:*

(1) For all $f \in wL^u_\Psi(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$, we have $\| L_x f \|_{wL^u_\Psi(\mathbb{R}^n)} \leq u(x) \| f \|_{wL^u_\Psi(\mathbb{R}^n)}$, where $L_x f(y) = f(y - x)$.

(2) If $f \in wL^u_\Psi(\mathbb{R}^n)$ and $f \neq 0$, then there exists a constant $B > 0$ (depends on f) such that

$$\frac{u(x)}{B} \leq \| L_x f \|_{wL^u_\Psi(\mathbb{R}^n)} \leq Bu(x).$$

Proof.

(1) Let $f \in wL^u_\Psi(\mathbb{R}^n)$ and $t, \epsilon > 0$. By using Lemma 2.4, we have

$$\begin{aligned}
\Psi(t)&\left|\left\{y \in \mathbb{R}^n: \frac{|u(y)L_x f(y)|}{u(x)(\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon)} > t\right\}\right| \\
&= \Psi(t)\left|\left\{y \in \mathbb{R}^n: \frac{|u(y)f(y-x)|}{u(x)(\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon)} > t\right\}\right| \\
&= \Psi(t)\left|\left\{v \in \mathbb{R}^n: \frac{|u(v+x)f(v)|}{u(x)(\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon)} > t\right\}\right| \\
&\leq \Psi(t)\left|\left\{v \in \mathbb{R}^n: \frac{|u(v)u(x)f(v)|}{u(x)(\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon)} > t\right\}\right| \\
&= \Psi(t)\left|\left\{v \in \mathbb{R}^n: \frac{|u(v)f(v)|}{(\| f \|_{wL^u_\Psi(\mathbb{R}^n)} + \epsilon)} > t\right\}\right| \\
&\leq 1
\end{aligned}$$

for $v := y - x$. Since $t > 0$ is arbitrary, we have

$$\sup_{t>0}\Psi(t)\left|\left\{y\in\mathbb{R}^n:\frac{|u(y)L_xf(y)|}{u(x)(\|f\|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)}>t\right\}\right|\leq 1.$$

This shows that $\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}\leq u(x)(\| f \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)$ for every $\epsilon>0$. Hence, we conclude that

$\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}\leq u(x)\ \| f \|_{wL_\Psi^u(\mathbb{R}^n)}$.

(2) Let $f\in wL_\Psi^u(\mathbb{R}^n)$ and $f\neq 0$, then there exists a constant $B_1>0$ such that $\| f \|_{wL_\Psi^u(\mathbb{R}^n)}\leq B_1$.

By Lemma 2.5 (1), we have $\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}\leq B_1u(x)$.

Now, for every $t,\epsilon>0$, we have

$$|\{v\in\mathbb{R}^n:\frac{|u(x)f(v)|}{\sup_{v\in\mathbb{R}^n}u(-v)(\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)}>t\}|\leq|\{v\in\mathbb{R}^n:\frac{|u(x)f(v)|}{u(-v)(\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)}>t\}|$$

$$\leq|\{v\in\mathbb{R}^n:\frac{|u(v+x)f(v)|}{\|L_xf\|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon}>t\}|$$

$$\leq|\{y\in\mathbb{R}^n:\frac{|u(y)f(y-x)|}{\|L_xf\|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon}>t\}|$$

$$=|\{y\in\mathbb{R}^n:\frac{|u(y)L_xf(y)|}{\|L_xf\|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon}>t\}|$$

for $y:=v+x$. So we obtain,

$$\Psi(t)|\{v\in\mathbb{R}^n:\frac{|u(x)f(v)|}{\sup_{v\in\mathbb{R}^n}u(-v)(\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)}>t\}|\leq\Psi(t)|\{y\in\mathbb{R}^n:\frac{|u(y)L_xf(y)|}{\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon}>t\}|\leq 1.$$

Since $t>0$ is arbitrary, we also obtain $\sup_{t>0}\Psi(t)|\{v\in\mathbb{R}^n:\frac{|u(x)f(v)|}{\sup_{v\in\mathbb{R}^n}u(-v)(\|L_xf\|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon)}>t\}|\leq 1$.

This show that

$$\frac{u(x)\ \| f \|_{wL_\Psi(\mathbb{R}^n)}}{\sup_{v\in\mathbb{R}^n}u(-v)}\leq\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}+\epsilon.$$

for every $\epsilon>0$. Choose $B:=\max\left\{B_1,\frac{\sup_{v\in\mathbb{R}^n}u(-v)}{\|f\|_{wL_\Psi^u(\mathbb{R}^n)}}\right\}$. Hence, we conclude that

$$\frac{u(x)}{B}\leq\| L_xf \|_{wL_\Psi^u(\mathbb{R}^n)}\leq Bu(x),$$

as desired.

Now, we are ready to present the necessary and sufficient conditions for the inclusion relation between two weighted weak Orlicz spaces $wL_{\Psi_1}^{u_1}(\mathbb{R}^n)$ and $wL_{\Psi_2}^{u_2}(\mathbb{R}^n)$ with regards to two Young functions Ψ_1,Ψ_2 and two weights u_1,u_2. The following theorem illustrates the corresponding result.

Theorem 2.6 *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \prec \Psi_2$ and u_1, u_2 are measurable functions such that $u_i(x+y) \le u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1,2$. Then the following three statements are equivalent:*

(1) $u_1 \preceq u_2$.
(2) $wL^{u_2}_{\Psi_2}(\mathbb{R}^n) \subseteq wL^{u_1}_{\Psi_1}(\mathbb{R}^n)$.
(3) There exists a constant $B > 0$ such that $\| f \|_{wL^{u_1}_{\Psi_1}(\mathbb{R}^n)} \le B \| f \|_{wL^{u_2}_{\Psi_2}(\mathbb{R}^n)}$, for every $f \in wL^{u_2}_{\Psi_2}(\mathbb{R}^n)$.

Proof.

Assume that (1) holds. Let f be an element of $wL^{u_2}_{\Psi_2}(\mathbb{R}^n)$. Since $\Psi_1 \prec \Psi_2$ and $u_1 \preceq u_2$, there exists constant $B_1, B_2 > 0$ such that $\Psi_1(t) \le \Psi_2(B_1 t)$ for all $t > 0$ and $u_1(x) \le B_2 u_2(x)$ for every $x \in \mathbb{R}^n$. Utilizing a similar argument presented in the proof of Theorem 2.3, we get

$$\| f \|_{wL^{u_1}_{\Psi_1}(\mathbb{R}^n)} \le B_1 \| f \|_{wL^{u_1}_{\Psi_2}(\mathbb{R}^n)} \le B_1 B_2 \| f \|_{wL^{u_2}_{\Psi_2}(\mathbb{R}^n)}.$$

As before, we obtain that (2) and (3) are equivalent. It thus leaves us to show that (3) implies (1). By assuming that (3) holds and considering Lemma 2.5, we have

$$\frac{u_1(x)}{B} \le \| L_x f \|_{wL^{u_1}_{\Psi_1}(\mathbb{R}^n)} \le B \| L_x f \|_{wL^{u_2}_{\Psi_2}(\mathbb{R}^n)} \le B u_2(x),$$

for every $x \in \mathbb{R}^n$. So, we obtain $u_1 \preceq u_2$.

It follows from Theorems 2.3 and 2.6 that there cannot be an inclusion relation between $wL^{u_1}_{p_1}(\mathbb{R}^n)$ and $wL^{u_2}_{p_2}(\mathbb{R}^n)$ for distinct values of p_1 and p_2. In spite of that, for a finite measure set X, we can obtain inclusion relation between $wL^{u_1}_{p_1}(X)$ and $wL^{u_2}_{p_2}(X)$, which will be discussed in the next section.

Regarding the fourth result, we will discuss sufficient conditions for Hölder's inequality in weighted weak Orlicz spaces in the following theorem. This theorem will be used to obtain inclusion relation between $wL^{u_1}_{p_1}(X)$ and $wL^{u_2}_{p_2}(X)$.

Theorem 3.1 *(Hölder's inequality) Let X be measurable set, Ψ_1, Ψ_2, Ψ_3 be Young functions and $u_1, u_2, u_3 : X \to \mathbb{R}$ be measurable functions such that $\Psi_1^{-1}(t)\Psi_2^{-1}(t) \le \Psi_3^{-1}(t)$ for every $t > 0$ and $u_3(x) \le u_1(x) u_2(x)$ for every $x \in X$. If $f_1 \in wL^{u_1}_{\Psi_1}(X)$ and $f_2 \in wL^{u_2}_{\Psi_2}(X)$, then $f_1 f_2 \in wL^{u_3}_{\Psi_3}(X)$ with*

$$\| f_1 f_2 \|_{wL^{u_3}_{\Psi_3}(X)} \le 2 \| f_1 \|_{wL^{u_1}_{\Psi_1}(X)} \| f_2 \|_{wL^{u_2}_{\Psi_2}(X)}.$$

Proof. Let f_i be elements of $wL^{u_i}_{\Psi_i}(X)$, $i = 1,2$. By Lemma 2.4, for every $k \in \mathbb{N}$ we have

$\Psi_1(t)|\{x \in X : \frac{|u_1(x) f_1(x)|}{(1+\frac{1}{k})\|f_1\|_{wL^{u_1}_{\Psi_1}(X)}} > t\}| \le 1$ and $\Psi_2(t)|\{x \in X : \frac{|u_2(x) f_2(x)|}{(1+\frac{1}{k})\|f_2\|_{wL^{u_2}_{\Psi_2}(X)}} > t\}| \le 1$

for every $t > 0$.

For each $x \in X$ and $k \in \mathbb{N}$, let

$$M(x,k) := \max(\Psi_1(\frac{|u_1(x) f_1(x)|}{(1+\frac{1}{k}) \| f_1 \|_{wL^{u_1}_{\Psi_1}(X)}}), \Psi_2(\frac{|u_2(x) f_2(x)|}{(1+\frac{1}{k}) \| f_2 \|_{wL^{u_2}_{\Psi_2}(X)}})).$$

From $\Psi_i(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}}) \leq M(x,k)$ and Lemma 1.3, we have

$$\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}} \leq \Psi_i^{-1}(\Psi_i(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}})) \leq \Psi_i^{-1}(M(x,k)), \text{ where } i = 1,2.$$

Hence $\prod_{i=1}^{2} \frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}} \leq \Psi_1^{-1}(M(x,k))\Psi_2^{-1}(M(x,k)) \leq \Psi_3^{-1}(M(x,k))$

and $\Psi_3(\prod_{i=1}^{2} \frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}}) \leq \Psi_3(\Psi_3^{-1}(M(x,k))) \leq M(x,k)$.

On the other hand, we have $M(x,k) \leq \sum_{i=1}^{2} \Psi_i(\frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}})$.

Therefore

$$\Psi_3(t)\left|\left\{x \in X: \prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{\left(1+\frac{1}{k}\right)\| f_i \|_{wL^{u_i}_{\Psi_i}(X)}} > t\right\}\right|$$

$$= \Psi_3\left(\prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|t_0}|f_i(x)|}{(1+\frac{1}{k})\| f_i \|_{wL^{u_i}_{\Psi_i}(X)}}\right)|\{x \in X: 1 > t_0\}|$$

$$\leq \Psi_3\left(\prod_{i=1}^{2} \frac{\sqrt{t_0}|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}}\right)|\{x \in X: 1 > t_0\}|$$

$$\leq \sum_{i=1}^{2} \Psi_i\left(\frac{\sqrt{t_0}|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}}\right)|\{x \in X: 1 > t_0\}|$$

where $t_0 := \frac{t(1+\frac{1}{k})^2\|f_1\|_{wL^{u_1}_{\Psi_1}(X)}\|f_2\|_{wL^{u_2}_{\Psi_2}(X)}}{|u_3(x)f_1(x)f_2(x)|}$.

Next, we also have

$$\Psi_i\left(\frac{\sqrt{t_0}|u_i(x)f_i(x)|}{(1+\frac{1}{k})\| f_i \|_{wL^{u_i}_{\Psi_i}(X)}}\right)|\{x \in X: 1 > t_0\}| = \Psi_i(t_i)|\{x$$

$$\in X: \left(\frac{|u_i(x)f_i(x)|}{\left(1+\frac{1}{k}\right)\| f_i \|_{wL^{u_i}_{\Psi_i}(X)}}\right)^2 > t_i^2\}|$$

$$= \Psi_i(t_i)|\{x \in X: \frac{|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}} > t_i\}| \leq 1$$

where $t_i = \frac{\sqrt{t_0}|u_i(x)f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL^{u_i}_{\Psi_i}(X)}}$, for $i = 1,2$.

So, we obtain $\Psi_3(t)|\{x \in X: \prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL_{\Psi_i}^{u_i}(X)}} > t\}| \le 2$.

On the other hand, we have

$$\begin{aligned}
\Psi_3(t)&\left|\left\{x \in X: \prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{\sqrt{2}\left(1+\frac{1}{k}\right)\| f_i \|_{wL_{\Psi_i}^{u_i}(X)}} > t\right\}\right| \\
&\le \sup_{t>0}\Psi_3(t)\left|\left\{x \in X: \prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{(1+\frac{1}{k})\| f_i \|_{wL_{\Psi_i}^{u_i}(X)}} > 2t\right\}\right| \\
&= \sup_{s>0}\Psi_3(\tfrac{s}{2})|\{x \in X: \textstyle\prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL_{\Psi_i}^{u_i}(X)}} > s\}| \\
&\le \sup_{s>0}\frac{1}{2}\Psi_3(s)\left|\left\{x \in X: \textstyle\prod_{i=1}^{2} \frac{\sqrt{|u_3(x)|}|f_i(x)|}{(1+\frac{1}{k})\|f_i\|_{wL_{\Psi_i}^{u_i}(X)}} > s\right\}\right| \le 1.
\end{aligned}$$

Since $t > 0$ is an arbitrary positive real number, we get

$$\sup_{t>0}\Psi_3(t)\left|\left\{x \in X: \frac{|u_3(x)f_1(x)f_2(x)|}{2(1+\frac{1}{k})^2 \| f_1 \|_{wL_{\Psi_1}^{u_1}(X)}\| f_2 \|_{wL_{\Psi_2}^{u_2}(X)}} > t\right\}\right| \le 1.$$

This shows that $\| f_1f_2 \|_{wL_{\Psi_3}^{u_3}(X)} \le 2(1+\frac{1}{k})^2 \| f_1 \|_{wL_{\Psi_1}^{u_1}(X)}\| f_2 \|_{wL_{\Psi_2}^{u_2}(X)}$ and this is true for every $k \in \mathbb{N}$. We can conclude that $\| f_1f_2 \|_{wL_{\Psi_3}^{u_3}(X)} \le 2 \| f_1 \|_{wL_{\Psi_1}^{u_1}(X)}\| f_2 \|_{wL_{\Psi_2}^{u_2}(X)}$, as desired.

Corollary 3.2 *Let* $X := B_a(r_0) \subseteq \mathbb{R}^n$ *for some* $a \in \mathbb{R}^n$ *and* $r_0 > 0$. *If* Ψ_1, Ψ_2 *are two Young functions,* $u_1, u_2: X \to \mathbb{R}$ *are measurable functions and there are a Young function* Ψ *and a weight* $0 < u(x) \le 1$ *for every* $x \in X$ *such that* $\Psi_1^{-1}(t)\Phi^{-1}(t) \le \Psi_2^{-1}(t)$ *for every* $t \ge 0$ *and* $u_1(x) \le u(x)u_2(x)$ *for every* $x \in X$, *then*

$$wL_{\Psi_1}^{u_1}(X) \subseteq wL_{\Psi_2}^{u_2}(X)$$

with $\| f \|_{wL_{\Psi_2}^{u_2}(X)} \le \frac{2}{\Psi^{-1}(\frac{1}{|B_a(r_0)|})} \| f \|_{wL_{\Psi_1}^{u_1}(X)}$ for $f \in wL_{\Psi_1}^{u_1}(X)$.

Proof. Since $0 < u(x) \le 1$ for every $x \in X$, we have $\| f \|_{wL_{\Psi_2}^{u_2}(X)} \le \| \frac{f}{u} \|_{wL_{\Psi_2}^{u_2}(X)}$. Let $f \in wL_{\Psi_1}^{u_1}(X)$, by Theorem 3.1 and choosing $g := \chi_{B_a(r_0)}$, we obtain

$$\begin{aligned}
\| f \|_{wL_{\Psi_2}^{u_2}(X)} &= \| f\chi_{B_a(r_0)} \|_{wL_{\Psi_2}^{u_2}(X)} \\
&= \| fg \|_{wL_{\Psi_2}^{u_2}(X)} \\
&\le \left\|\frac{fg}{u}\right\|_{wL_{\Psi_2}^{u_2}(X)} \\
&\le 2\left\|\frac{g}{u}\right\|_{wL_{\Psi}^{u}(X)} \| f \|_{wL_{\Psi_1}^{u_1}(X)} \\
&= \frac{2}{\Psi^{-1}(\frac{1}{|B_a(r_0)|})} \| f \|_{wL_{\Psi_1}^{u_1}(X)}.
\end{aligned}$$

This shows that $wL^{u_1}_{\Psi_1}(X) \subseteq wL^{u_2}_{\Psi_2}(X)$.

Fifth, regarding the last result in this paper, we shall now discuss the inclusion properties of weighted weak Lebesgue spaces $wL^{u_1}_{p_1}(X)$ and $wL^{u_2}_{p_2}(X)$ with respect to distinct values of p_1 and p_2 as well as u_1 and u_2.

Corollary 3.3 *Let* $X := B_a(r_0)$ *for some* $a \in \mathbb{R}^n$ *and* $r_0 > 0$. *If* $1 \leq p_2 < p_1 < \infty$ *and* $u_1, u_2 : X \to \mathbb{R}$ *are measurable functions such that* $u_1(x) \leq u_2(x)$ *for every* $x \in X$, *then* $wL^{u_1}_{p_1}(X) \subseteq wL^{u_2}_{p_2}(X)$.

Proof. Let $\Psi_1(t) := t^{p_1}, \Psi_2(t) := t^{p_2}$, $\Psi(t) := t^{\frac{p_1 p_2}{p_1 - p_2}}$ for every $t \geq 0$. Since $1 \leq p_2 < p_1 < \infty$, we have $\frac{p_1 p_2}{p_1 - p_2} > 1$. Thus, Ψ_1, Ψ_2, and Ψ are Young functions. Now, define $u(x) = \frac{u_1(x)}{u_2(x)}$ for every $x \in X$. Note that, by operating the definition of Ψ^{-1} and Lemma 1.3, we obtain

$$\Psi_1^{-1}(t) = t^{\frac{1}{p_1}}, \qquad \Psi_2^{-1}(t) = t^{\frac{1}{p_2}}, and \;\; \Psi^{-1}(t) = t^{\frac{p_1 - p_2}{p_1 p_2}}.$$

Moreover, $\Psi_1^{-1}(t)\Psi^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{p_1 - p_2}{p_1 p_2}} = t^{\frac{1}{p_2}} = \Psi_2^{-1}(t)$ and $u_1(x) = \frac{u_1(x)}{u_2(x)} u_2(x) = u(x)u_2(x)$. Consequently, Corollary 3.2 implies $\| f \|_{wL^{u_2}_{p_2}(X)} \leq 2|B_a(r_0)|^{\frac{p_1 - p_2}{p_1 p_2}} \| f \|_{wL^{u_1}_{p_1}(X)}$, and therefore we can conclude that $wL^{u_1}_{p_1}(X) \subseteq wL^{u_2}_{p_2}(X)$.

Remark. Theorem 3.1 potentially plays an essential role in the study of the boundness of the Stein-Weiss operator on weighted Orlicz spaces because this operator is a solution of a differential equation.

4. Conclusion

To conclude this paper, we have reached our main goal, which is to show the inclusion properties of weighted weak Lebesgue spaces and weighted weak Orlicz spaces. In short, the inclusion properties of weighted weak Orlicz spaces are a generalization of the inclusion properties of weak Orlicz spaces and the inclusion properties of weighted weak Lebesgue spaces as elaborated in. In proving our results, we utilized the norm of characteristic function on $\mathbb{R}^n$ and estimate the norm of the translation functions in $\mathbb{R}^n$. From Theorem 2.3 and Corollary 2.16 in, we have that the inclusion properties of weighted Orlicz spaces and the inclusion properties of weighted weak Orlicz spaces are equivalent to the same condition, namely $u_1 \preceq u_2$. Consequently, the inclusion properties of weighted Lebesgue spaces are also equivalent to the inclusion properties of weighted weak Lebesgue spaces.

Furthermore, from Lemma 1.1, Theorem 1.2, and Theorem 2.6, we also have the following inclusion relations

$$\begin{array}{ccc} L^{u_2}_{\Psi_2} & \to & L^{u_1}_{\Psi_1} \\ \downarrow & \searrow & \downarrow \\ wL^{u_2}_{\Psi_2} & \to & wL^{u_1}_{\Psi_1} \end{array}$$

for $\Psi_1 \prec \Psi_2$ and $u_1 \preceq u_2$, where the arrows mean 'contained in' or 'embedded into'. In Theorem 2.6, we also assume that $u_i(x + y) \leq u_i(x) \cdot u_i(y)$ for every $x, y \in \mathbb{R}^n$, where $i = 1,2$. It can be interesting to investigate whether this condition can be removed or not. In addition, this study hopefully motivates further investigation on inclusion properties and boundness of generalized H *ö* lder inequality on weighted Orlicz-Morrey spaces.

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