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NECESSARY CONDITION FOR BOUNDEDNESS OF STEIN-WEISS OPERATOR ON ORLICZ SPACES

SITI FATIMAH[1,*], SOFIHARA AL HAZMY[2], AL AZHARY MASTA[1]

[1]Department of Mathematics Education, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi 229, Bandung 40154, Indonesia
[2]Faculty of Military Mathematics and Science, Universitas Pertahanan Republik Indonesia, Kawasan IPSC Sentul, Sukahati, Bogor, Jawa Barat 16810
*Corresponding Author: sitifatimah@upi.edu

Abstract

This study aims to develop necessary conditions for the boundedness of Stein-Weiss Operator on Orlicz spaces. It is well known that the Orlicz space is the generalization of the Lebesque space. In particular, by considering power function as Young's function, the Orlicz space is a Lebesgue space itself. Orlicz space and several operators in the space have been studied intensively by several researchers. In this study, we find the necessary condition for boundedness of Stein-Weiss operator on Orlicz spaces. The technique to achieve our purpose is by substituting dilation of radial function on a ball into inequality of boundedness assumption, and some basic properties of Young's function. Since the dilation is dynamic, we will get the result as inequality in one parameter. Furthermore, Young's function of range times certain power function is dominated by Young's function of domain. As a discussion, we try to find several examples that satisfy the inequalities. The most important, the result shows that there is a significant factor to see the boundedness of the Stein-Weiss operator on Orlicz space.

Keywords: Boundedness, Orlicz spaces, Stein-Weiss operator.

1. Introduction

There are many authors have studied further about quasilinear operators on Orlicz spaces. Nakai [1], Hashimoto et al. [2] already obtained sufficient conditions for boundedness of generalized fractional integral operators on an integrated version of Orlicz spaces. Guliyev and Deringoz [3] derived the necessary and sufficient condition for boundedness of fractional maximal operator on generalized Orlicz-Morrey spaces. Hasto [4] investigates the properties of a maximal operator on generalized Orlicz spaces.

One of the incredible results is obtained by Cianchi [5]. He already found necessary and sufficient conditions for boundedness of several classical operators on Orlicz spaces, and the results are very sharp. In Theorem 1, 2, 3 [5], he found the necessary and sufficient condition for weak and strong boundedness of maximal operator, fractional integral operator, and singular integral on Orlicz spaces.

Also, Sawano et al. [6] have found the results of hard work for inequalities of fractional operators on Orlicz-Morrey spaces. Those are available in Theorem 2.8, 2.9, 2.10, and corollary 2.11 [6].

As the beginning of our research, we want to exploit every inch of properties of a quasilinear operator, we called the operator as Stein-Weiss operator. Stein-Weiss operator was first introduced by Stein-Weiss [7]. They prove that the operator:

$$T_{\alpha,\lambda,\beta} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{\alpha}|x-y|^{n-\lambda}|x|^{\beta}}\, dy$$

is bounded from Lebesgue space L^p to Lebesgue space L^q, which α, λ, β satisfy that shown in Eqs. (1) and (2).

$$0 < \alpha < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq 0 \tag{1}$$

and

$$\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta-\lambda}{n} \tag{2}$$

In fact, if Eq. (1) is given, then Eq. (2) is necessary and sufficient condition for boundedness of $T_{\alpha,\lambda,\beta}$. We called operator $T_{\alpha,\lambda,\beta}$ as Stein-Weiss operator.

In this paper, we want to find necessary and sufficient condition for boundedness of Stein-Weiss operator on Orlicz spaces. Before defining Orlicz space, we need to know several facts about Young's function. For the definitions and properties of several facts related to Orlicz spaces, we use some references [8-11]. Recall the definition of Lebesgue space,

$$L^p(\mathbb{R}^n) = \left\{ f : f \text{ measurable, and } \int_{\mathbb{R}^n} |f(x)|^p\, dx < \infty \right\}.$$

Lebesgue space is complete space with norm

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx \right)^{\frac{1}{p}}.$$

There are several generalizations of Lebesgue space, one of them is Orlicz space. To define Orlicz space, we require a few support definitions.

Define a function $\phi : [0,\infty) \to [0,\infty)$ is Young's function if and only if ϕ is convex, $\phi(0) = 0$, and $\lim_{x\to\infty} \phi(x) = \infty$.

Since ϕ is convex on open set $(0,\infty)$ then it is continuous on $(0,\infty)$. Since ϕ also map $[0,\infty) \to [0,\infty)$ and $\phi(0) = 0$, then it is continuous on $[0,\infty)$. Also, ϕ is strictly increasing on its support $\{x \geq 0 : \phi(x) > 0\}$.

Next, we recall the definition of the inverse of Young's function. For any Young's function ϕ, its inverse is given by

$$\phi^{-1}(y) = \inf\{s \geq 0 : \phi(s) \geq y\}.$$

Some of basic properties of ϕ^{-1} is shown in Eq. (3).

$$\phi^{-1}\big(\phi(x)\big) \leq x \leq \phi\big(\phi^{-1}(x)\big) \tag{3}$$

In fact, for every $x \in [0,\infty)$, we have $x \in \{s \geq 0 : \phi(s) \geq \phi(x)\}$, therefore $\phi^{-1}\big(\phi(x)\big) \leq x$. On the other hand, by properties of infimum, there exists some sequences (s_n) such that $s_n \to \phi^{-1}(x)$ and $\phi(s_n) \geq x$ for every n. The continuity of ϕ implies $\phi\big(\phi^{-1}(x)\big) = \lim_{n\to\infty} \phi(s_n) \geq \lim_{n\to\infty} x = x$.

Since convex function ϕ is bijection on its support, then equality of Eq. (3) does hold on to support of ϕ. That is, if $\phi(x) > 0$, then

$$\phi^{-1}\big(\phi(x)\big) = x = \phi\big(\phi^{-1}(x)\big).$$

Furthermore, $\varphi^{-1}(y) = 0$ if and only if $y = 0$. Also, if $c > 0$ then shown in Eq. (4)

$$\{x > 0 : \phi(x) \leq c\} = \{x > 0 : x \leq \phi^{-1}(c)\} \tag{4}$$

To prove Eq. (4), let A be the left-hand side, and B for the other one. Suppose $x \in A$. Since $c > 0$, then $\phi^{-1}(c) > 0$. If $\phi(x) = 0$ then $x \notin P = \{s \geq 0 : \phi(s) \geq c\}$, so x is the lower bound of P, therefore $x \leq \inf P = \phi^{-1}(c)$. If $\phi(x) > 0$, then $\phi^{-1}(\phi(x)) = \phi(\phi^{-1}(x)) = x$, means $x \in B$. On the other hand, suppose $x \in B$, then $x \leq \phi^{-1}(c)$Therefore, $\phi(x) \leq \phi(\phi^{-1}(c))$. Since $\phi^{-1}(c) > 0$ and ϕ is bijection on its support, then $\phi(\phi^{-1}(c)) = c$. This concludes that $x \in A$.

Next, we observe that,

$$\phi^{-1}(Ky) \leq K\phi^{-1}(y).$$

For any $K \geq 1$. In fact, suppose that $K \geq 1$, by definition of inverse, we have,

$$\phi^{-1}(Ky) = \inf\left\{s \geq 0 : \frac{\phi(s)}{K} \geq y\right\}.$$

Since ϕ is convex function, we have $\frac{\phi(s)}{K} \geq \phi\left(\frac{s}{K}\right)$. Hence,

$$\left\{s \geq 0 : \phi\left(\frac{s}{K}\right) \geq y\right\} \subseteq \left\{s \geq 0 : \frac{\phi(s)}{K} \geq y\right\}.$$

Which is equivalent with

$$\{Ks \geq 0: \phi(s) \geq y\} \subseteq \left\{s \geq 0: \frac{\phi(s)}{K} \geq y\right\}.$$

By taking infimum, the left-hand side is greater than the right-hand side. That is Eq. (5),

$$\phi^{-1}(Ky) \leq K\phi^{-1}(y). \tag{5}$$

for any $K \geq 1$. This completes our observation.

The properties Eqs. (4) and (5) will be useful later. Now, we recall the definition of Orlicz spaces. Suppose that ϕ is Young's function, Orlicz space L^{ϕ} is given by

$$L^{\phi}(\mathbb{R}^n) = \left\{f: f measurable, there\ exists\ c > 0\ such\ that\ \int_{\mathbb{R}^n} \phi(c|f(x)|)\ dx < \infty\right\}$$

Obviously, $L^{\phi}(\mathbb{R}^n)$ is vector space. There are two famous norms which is usually defined on $L^{\phi}(\mathbb{R}^n)$. The first is Luxemburg's norm, and the other is Orlicz's norm. For defining Orlicz's norm, we need to define the conjugate of a Young's function. Let ϕ is a Young's function, its conjugate is given by

$$\phi^*(x) = \sup\{tx - \phi^*(t): t \geq 0\}.$$

Next, for every $f \in L^{\phi}(\mathbb{R}^n)$, Luxemburg's norm of f is given by,

$$\|f\|_{L^{\phi}(\mathbb{R}^n)} = \inf\left\{b > 0: \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{b}\right) dx \leq 1\right\},$$

and its Orlicz's norm is given by,

$$\|f\|_{L^{\tilde{\phi}}(\mathbb{R}^n)} = \mathrm{Sup}\left\{\int_{\mathbb{R}^n} f(x)g(x)\ dx: \|f\|_{L^{\phi^*}(\mathbb{R}^n)} \leq 1\right\},$$

These two norms are Eq. (6), in fact,

$$\|f\|_{L^{\tilde{\phi}}(\mathbb{R}^n)} \leq \|f\|_{L^{\phi}(\mathbb{R}^n)} \leq 2\|f\|_{L^{\tilde{\phi}}(\mathbb{R}^n)}. \tag{6}$$

Especially for $\phi(t) = t^p$, we have the Orlicz norm becomes Lebesgue norm, that is $\|f\|_{L^{\phi}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$. Remark. The first inequality of Eq. (6) is obtained from Hölder's Inequality, for the further study about Hölder inequalities on Orlicz's spaces, we refer to Ifronika et al. [12] and Masta et al. [13]. Also, the generalized of Orlicz spaces can be found in [14, 15].

Since many authors [1, 2, 3, 5, 6, 16] have studied fractional integral operators, so we are taking interest in Stein-Weiss Operator as a generalization of fractional integral operator. The novelties of this study were the necessary condition for the boundedness of Stein-Weiss operator.

2. Method

First, we will assume that the Stein-Weiss Operator is bounded on Orlicz spaces. Second, we substitute specific function

$$f(x) = |x|\chi_{B(0,1)}(x)$$

and its dilation

$$g_t f(x) = f(tx).$$

into the inequality

$$\|T_{\alpha,\lambda,\beta}f\|_{L^{\psi}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\phi}(\mathbb{R}^n)}$$

Third, we solve the inequality above to get some relation between ϕ and ψ, which is it will be our necessary condition.

Since *f* is our trial function, we have to check that *f* is an element of Orlicz space. In fact, for any Young's function ϕ,

$$\|f\|_{L^{\phi}(\mathbb{R}^n)} = \inf\left\{b > 0\colon \int_{B(0,1)} \phi\left(\frac{|x|^{\alpha}}{b}\right) dx \le 1\right\}$$

Since,

$$\inf\left\{b > 0\colon \int_{B(0,1)} \phi\left(\frac{|x|^{\alpha}}{b}\right) dx \le 1\right\} = C_n \int_0^1 \phi\left(\frac{r^{\alpha}}{b}\right) r^{n-1}\, dr \le C_n\phi\left(\frac{1}{b}\right)\int_0^1 r^{n-1}\, dr = C_n\phi\left(\frac{1}{b}\right)$$

It means that,

$$\left\{b > 0\colon C_n\phi\left(\frac{1}{b}\right) \le 1\right\} \subseteq \left\{b > 0\colon \int_{B(0,1)} \phi\left(\frac{|x|^{\alpha}}{b}\right) dx \le 1\right\}$$

Therefore,

$$\|f\|_{L^{\phi}(\mathbb{R}^n)} \le \inf\left\{b > 0\colon C_n\phi\left(\frac{1}{b}\right) \le 1\right\} < \infty.$$

This shows that f is in Orlicz space. Similarly, we also have g_t is element of the same Orlicz space for every $t > 0$.

3. Result and Discussion

Stein-Weiss [7] have proven that Stein-Weiss operator is bounded in Lebesgue space if certain condition Eq. (1) and Eq. (2) are satisfied. In fact, they proved that if Eq. (1) is given then Eq. (2) is necessary and sufficient condition for the boundedness in Lebesgue space. We want to develop their results in Orlicz space. In this section, we prove the necessary condition for boundedness Stein-Weiss Operator on Orlicz spaces. The following theorem is the main result of this paper.

Theorem 1. suppose that ϕ and ψ are Young's function. Let $\alpha, \beta \ge 0$. If $T_{\alpha,\lambda,\beta}$ is bounded from $L^{\phi} \to L^{\psi}$, then shown in Eq. (7).

$$t^{\frac{\alpha+\beta-\lambda}{n}}\phi^{-1}(t) \lesssim \psi^{-1}(t) \tag{7}$$

for every $t > 0$.

Proof. Suppose that $T_{\alpha,\lambda,\beta}$ is bounded from $L^{\phi} \to L^{\psi}$, then

$$\|T_{\alpha,\lambda,\beta}f\|_{L^{\psi}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\phi}(\mathbb{R}^n)}.$$

for every $f \in L^{\phi}(\mathbb{R}^n)$. Since $f(x)|x|\chi_{B(0,1)}(x)$ is element of Orlicz space L^{ϕ}, then $g_t(x) = f(tx$ is also element of Orlicz space $L^{\phi}(\mathbb{R}^n)$. Notice that

$\int_{\mathbb{R}^n} \phi\left(\frac{|g_t(x)|}{b}\right) dx \;= \int_{\mathbb{R}^n} \phi\left(\frac{|tx|^\alpha \chi_{B(0,1)}(tx)}{b}\right) dx = t^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{|x|^\alpha \chi_{B(0,1)}(x)}{b}\right) dx =$
$t^{-n} \int_{B(0,1)} \phi\left(\frac{|x|^\alpha}{b}\right) d = a_n t^{-n} \int_0^1 \phi\left(\frac{r^\alpha}{b}\right) r^{n-1} dr \leq a_n t^{-n} \phi\left(\frac{1}{b}\right).$

Then we have,

$$\|g_t\|_{L^\phi} \leq \inf\left\{b > 0: a_n t^{-n} \phi\left(\frac{1}{b}\right) \leq 1\right\}$$
$$\leq \inf\left\{b > 0: \phi\left(\frac{1}{b}\right) \leq a_n^{-1} t^n\right\}$$
$$= \inf\left\{b > 0: \frac{1}{b} \leq \phi^{-1}(a_n^{-1} t^n)\right\}$$
$$= \frac{1}{\phi^{-1}(a_n^{-1} t^n)}.$$

The first equality is obtained from Eq. (4). On the other hand,

$$T_{\alpha,\lambda,\beta} g_t(x) = \int_{\mathbb{R}^n} \frac{g_t(y)}{|y|^\alpha |x-y|^{n-\lambda} |x|^\beta}\, dy$$
$$= \int_{\mathbb{R}^n} \frac{f(ty)}{|y|^\alpha |x-y|^{n-\lambda} |x|^\beta}\, dy$$
$$= \int_{\mathbb{R}^n} \frac{f(u) t^{-n}}{|t^{-1}u|^\alpha |x - t^{-1}y|^{n-\lambda} |x|^\beta}\, dy$$
$$= \int_{\mathbb{R}^n} \frac{f(u) t^{-n}}{t^{-\alpha} |u|^\alpha t^{\lambda-n} |tx-u|^{n-\lambda} t^{-\beta} |tx|^\beta}\, dy$$
$$= \; t^{\alpha+\beta-\lambda} T_{\alpha,\lambda,\beta} f(tx).$$

Therefore,

$$\left\|T_{\alpha,\lambda,\beta} g_t\right\|_{L^{\psi}(\mathbb{R}^n)} = t^{\alpha+\beta-\lambda} \inf\left\{b > 0: t^{-n} \int_{\mathbb{R}^n} \psi\left(\frac{\left|T_{\alpha,\lambda,\beta} f(x)\right|}{b}\right) dx \leq 1\right\}.$$

Notice that,

$$T_{\alpha,\lambda,\beta} f(x) = \int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda} |x|^\beta}\, dy.$$

And,

$$\int_{\mathbb{R}^n} \psi\left(\frac{\left|T_{\alpha,\lambda,\beta} f(x)\right|}{b}\right) dx \geq \int_{B\left(0,\frac{1}{2}\right)} \psi\left(b^{-1} \int_{B(0,1)} \frac{1}{|x-y|^{n-\lambda} |x|^\beta}\, dy\right) dx.$$

Also, for$x \in B\left(0, \frac{1}{2}\right)$ we have $B\left(x, \frac{1}{2}\right) \subset B(0,1)$. Therefore,

$$\int_{\mathbb{R}^n} \psi\left(\frac{\left|T_{\alpha,\lambda,\beta} f(x)\right|}{b}\right) dx \geq \int_{B\left(0,\frac{1}{2}\right)} \psi\left(b^{-1} |x|^{-\beta} \int_{B\left(x,\frac{1}{2}\right)} \frac{1}{|x-y|^{n-\lambda}}\, dy\right) dx$$

$$= \int_{B\left(0,\frac{1}{2}\right)} \psi\left(\frac{a_n|x|^\wedge\beta}{2^\lambda \lambda b}\right) dx$$
$$= a_n \int_0^{\frac{1}{2}} \psi\left(\frac{a_n r^\beta}{2^\lambda \lambda b}\right) \cdot r^{n-1}\, dr$$
$$\geq a_n \int_{\frac{1}{4}}^{\frac{1}{2}} \psi\left(\frac{a_n r^\beta}{2^\lambda \lambda b}\right) \cdot r^{n-1}\, dr$$
$$\geq a_n 4^{-n} \psi\left(\frac{a_n}{4^\beta 2^\lambda \lambda b}\right).$$

So, we have,

$$\left\|T_{\alpha,\lambda,\beta} g_t\right\|_{L^{\psi}(\mathbb{R}^n)} \geq t^{\alpha+\beta-\lambda} \inf\left\{b > 0 : t^{-n} a_n 4^{-n} \psi\left(\frac{a_n}{4^\beta 2^\lambda \lambda b}\right) \leq 1\right\}$$
$$= t^{\alpha+\beta-\lambda} \inf\left\{b > 0 : \psi\left(\frac{a_n}{4^\beta 2^\lambda \lambda b}\right) \leq a_n^{-1} 4^n t^n\right\}$$
$$= t^{\alpha+\beta-\lambda} \inf\left\{b > 0 : \frac{a_n}{4^\beta 2^\lambda \lambda b} \leq \psi^{-1}(a_n^{-1} 4^n t^n)\right\}$$
$$= \frac{a_n t^{\alpha+\beta-\lambda}}{4^\beta 2^\lambda b \psi^{-1}(a_n^{-1} 4^n t^n)}.$$

Since $T_{\alpha,\lambda,\beta}$ is bounded from $L^\varphi \to L^\psi$, then

$$\frac{a_n t^{\alpha+\beta-\lambda}}{4^\beta 2^\lambda b \psi^{-1}(a_n^{-1} 4^n t^n)} \leq \frac{1}{\phi^{-1}(a_n^{-1} t^n)}$$

for every $r, t > 0$. This is equivalent with

$$t^{\alpha+\beta-\lambda} \phi^{-1}(a_n^{-1} t^n) \lesssim \psi^{-1}(a_n^{-1} 4^n t)$$

By replacing t with $(a_n t)^{\frac{1}{n}}$ we have

$$t^{\frac{\alpha+\beta-\lambda}{n}} \phi^{-1}(t) \lesssim \psi^{-1}(4^n t).$$
$$t^{\frac{\alpha+\beta-\lambda}{n}} \phi^{-1}(t) \lesssim \psi^{-1}(t).$$

for every $t > 0$.

Next, our discussion is to find several examples that satisfy (7). Suppose ϕ and ψ are power function, say $\phi(t) = t^p$ and $\psi(t) = t^q$, we have the inverses are $\phi^{-1}(t) = t^{\frac{1}{p}}$ and $\psi^{-1}(t) = t^{\frac{1}{q}}$. Then,

$$t^{\frac{\alpha+\beta-\lambda}{n}} t^{\frac{1}{p}} \lesssim t^{\frac{1}{q}}$$

if and only if

$$t^{\frac{\alpha+\beta-\lambda}{n} + \frac{1}{p} - \frac{1}{q}} \lesssim 1$$

if and only if Eq. (8),

$$\frac{1}{q} = \frac{\alpha+\beta-\lambda}{n} + \frac{1}{p}. \tag{8}$$

This result shows that parameters $\alpha, \beta, \lambda, p, q$ that satisfy (8) are the necessary condition for boundedness of $T_{\alpha,\lambda,\beta}$ from L^p to L^q.

It is not always obvious to have a function ψ that satisfy (7). For example, consider

$$\phi_{\exp}(t) = e^t - 1.$$

Then $\phi_{\exp}^{-1}(t) = \ln(1+t)$. So, we need to find a Young's function ψ that satisfy

$$t^{\frac{\alpha+\beta-\lambda}{n}} \ln(1+t) \lesssim \psi^{-1}(t).$$

Since we need a Young's function ψ, then ψ^{-1} must be concave, and $\psi^{-1}(0) = 0$.

Sometimes we know the function ψ exists but cannot be described concretely. For example, consider Young's function Φ in [2] (Corollary 2.2). Suppose that,

$$\Phi(t) = t\ln(t+c) \text{ for some c} > e.$$

Since $\Phi'(t) > 0$ for $t \geq 0$, then it is strictly increasing, so the inverse is existed. Also, $\Phi''(t) > 0$, then Φ is convex. Let, $\alpha + \beta = 0$, so ψ must be satisfy

$$t^{-\frac{\lambda}{n}} \Phi^{-1}(t) \lesssim \psi^{-1}(t).$$

By choosing ψ as a function that satisfy Eq. (9)

$$\psi^{-1}(t) = t^{-\frac{\lambda}{n}} \Phi^{-1}(t), \tag{9}$$

We can find the second derivative of ψ^{-1} in Φ and Φ^{-1}, which is ψ^{-1} can be concave for certain λ, this implies ψ is convex for certain λ.

Stein-Weiss operator is the solution of fractional Laplacian equations. In particular, Laplace equation is the special form. In Technology, Heat equation is one of the examples that conceive the Laplace equation. This result is one of the properties of the solution of fractional Laplacian equations.

4. Further Results and Concluding Remarks

The necessary condition for boundedness of Stein-Weiss operator on Orlicz space is already presented in Theorem 1. Properties Eq. (7) give a little picture that leads us to appropriate Young's function. Several examples such as power function can be obviously matched with Eq. (7), although some suitable function cannot be described concretely, such as ψ on the Eq. (9), we know the function is suitable. Since the Stein-Weiss Operator is the solution of fractional Laplacian equations, Eq. (7) is one of the properties. A similar method should be work for finding the boundedness of such an operator on Orlicz-Morrey spaces.

As a recommendation for further study, we refer to [1-7] for finding various techniques to develop our results.

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References

1. Nakai, E. (2001). On generalized fractional integrals in orlicz spaces on spaces of homogeneous type. *Scientiae Mathematicae Japonicae Online*, 4, 901-915.
2. Hashimoto, D.; Ohno, T.; and Shimomura, T. (2019). Boundedness of generalized fractional integral operators on Orlicz spaces near L^1 over metric measure spaces. *Czechoslovak Mathematical Journal*, 69(1), 207-223.
3. Guliyev, V.S.; and Deringoz, F. (2014). Boundedness of fractional maximal operator and its commutators on generalized orlicz-morrey spaces. *Complex Analysis and Operator Theory,* 9(6), 1249-1267.
4. Hästö, P.A. (2015). The maximal operator on generalized Orlicz spaces. *Journal of Functional Analysis*, 269(12), 4038-4048.
5. Cianchi, A. (1999). Strong and weak type inequalities for some classical operators in Orlicz spaces. *Journal of the London Mathematical Society*, 60(1), 187-202.
6. Sawano, Y.; Sugano, S.; and Tanaka, H. (2011). Orlicz-Morrey spaces and fractional operators. *Potential Analysis,* 36(4), 517-556
7. Kassymov, A.; Ruzhansky, M.; and Suragan, D. (2019). Hardy–Littlewood–Sobolev and Stein–Weiss inequalities on homogeneous Lie groups. *Integral Transforms and Special Functions*, 30(8), 643-655.
8. Darst, R.B.; and Zink, R.E. (1995/1996). A note on the definition of an orlicz space. *Real Analysis Exchange*, 21(1), 356-362.
9. Li, Y.-C.; and Yeh, C.-C. (2010). Some characterizations of convex functions. *Computers and Mathematics with Applications*, 59(1), 327-337.
10. Masta, A.A.; Gunawan, H.; and Budhi, W.S. (2016). Inclusion properties of Orlicz and weak Orlicz spaces. *Journal of Mathematics and Fundamental Sciences*, 48(3), 193-203.
11. Welland, R. (1966). Inclusion relations among orlicz spaces. *Proceedings of the American Mathematical Society*. 17(1), 135-139.
12. Ifronika; Masta, A.A.; Nur, M.; and Gunawan, H. (2018). Generalized hölder's inequality in orlicz space. *Proceedings of the Jangjeon Mathematical Society.* Bandung, Indonesia, 25-34.
13. Masta, A.A.; Fatimah, S.; Arsisari, A.; Putra, Y.Y.; and Apriani, F. (2019). Sufficient and necessary conditions for generalized hölder inequality in p-summable sequence spaces. Proceedings of the Asian Mathematics Conference. *Journal of Physic: Conference Series,*1280.
14. Masta, A.A.; Gunawan, H.; and Setya-Budhi, W. (2017). An inclusion property of Orlicz-Morrey spaces. *Proceedings of the Asian Mathematics Conference*. Bali, Indonesia.
15. Masta, A.A.; Fatimah, S.; and Taqiyuddin, M. (2019). Third version of weak orlicz–morrey spaces and its in-clusion properties. *Indonesian Journal of Science and Technology*, 4(2), 257-262.
16. Hazmy, S. A.; Setia-Budhi, W.; and Soeharyadi, Y. (2019). Boundedness of modified fractional integral operators on morrey spaces. *Journal of Physics: Conference Series*, Volume 1127, The *6th International Conference on Mathematics and Natural Sciences* 2–3 November 2016, Bandung, Indonesia.