Journal of Engineering Science and Technology
Vol. 13, No. 4 (2018) 893 - 908

IDENTIFICATION OF SECOND SPECTRUM OF A TIMOSHENKO BEAM USING DIFFERENTIAL TRANSFORM METHOD

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Abstract

In the present work, the vibration characteristics of a Timoshenko beam are examined using Differential Transform Method (DTM) with the end conditions hinge-hinge, fix-hinge, fix-fix and fix-free. The frequencies are computed for the beam with the length to depth ratio (*L/H*) of 2, 3, 5, 10 and 20. The results obtained from the DTM are compared with exact values and finite element results. DTM analysis shows that the second spectrum of frequency is observed in beams which are comparatively thick ($L/H \leq 10$), with hinge-hinge end condition. Eigenfunctions for both frequency spectra are derived using this method. The convergence analysis of frequencies of a Timoshenko beam obtained using DTM is also presented. The second set of frequencies are not reflected in the vibration of beams with end conditions other than hinge-hinge. The mode shapes are presented for the pure shear mode and the first three bending and shear modes of hinge-hinge supported Timoshenko beam.

Keywords: Timoshenko beam, Second spectrum, Differential transform method, Bending spectrum, Shear spectrum.

1. Introduction

Several mathematical models are available to study the vibration characteristics of beams. Among them, the classical Euler-Bernoulli theory, which neglects shear strain and rotary inertia and the Timoshenko beam theory (TBT) which considers both shear strain and rotary inertia are quite commonly used in practice. The former one is accurate to model slender beam vibrations, but becomes erroneous

Nomenclatures

A	Sectional area of beam, m
A_n	Amplitude of n^{th} mode
D_1, D_2	Arbitrary constants
E	Young's Modulus, N/m^2
$f(x)$	Function of x
$F[k]$	Differential Transform of $f(x)$
G	Shear Modulus, N/m^2
I	Moment of Inertia, m^4
K_s	Shear Correction factor
k	Number of terms taken in the polynomial
L	Length of beam, m
L/H	Length to Depth ratio
n	Mode number
N	The value of k for the convergence
Q	Dimensionless parameter for material properties
R	Dimensionless parameter for geometry of beam
S	Rotation Amplitude, rad
t	Time, s
w	Deflection of the beam, m
$W(x)$	Modeshape for displacement
$\overline{W}$	Dimensionless displacement
x	Spatial coordinate
$\frac{dw}{dx}-\psi$	Shear Strain

Greek Symbols

ξ	Dimensionless spatial co-ordinate (x/L)
ρ	Density of the material, kg/m^3
ψ	Slope of deflection curve
Ω	Non-dimensional frequency
ω	Frequency of vibration, rad/s

Abbreviations

DTM	Differential Transform Method
FEM	Finite Element Method
TBT	Timoshenko Beam Theory

while modelling thick beams. On the other hand, latter is accurate in both thin and thick beam conditions. However, the Timoshenko beam model is difficult to solve analytically even for simple end conditions, such as fix-free, fix-fix, etc. Hence, engineers employ numerical techniques to solve Timoshenko beam model for most practical applications. They are successfully used to model aircraft wing vibrations, turbine blade vibrations, etc.

The vibration characteristics of beams under various structural conditions using TBT have been examined by researchers in the past using analytical and numerical methods. It has been shown by several investigators that the Timoshenko theory for straight beams offers two spectra for natural frequencies, the flexural (bending) dominated first spectra and the shear dominated second spectra. Even though the second spectrum is only of academic interest at present, the authors assume that the knowledge of the frequency spectrum of Timoshenko beam is essential in the case of the analysis of rotating beam wherein the shear effect and the rotary inertia cannot be neglected and for the design of aircraft components which are prone to instabilities like flutter which is attributed to the coupling of flexural and torsional mode of the structure. However, the exact implications of these frequency curves on real life applications are yet to be explored with further studies.

Analytical solutions for simple end conditions are available in the literatures. The existence of the second spectra was first reported by Traill-Nash and Collar [1] for the cases of hinge-hinge and free-free end conditions. Later, Anderson and Dolph [2, 3] confirmed the existence of the second spectra for the hinge-hinge beam. Tobe and Sato [4] investigated the existence of the second spectra for the cantilever beam and experienced difficulty in classifying the frequencies. Abbas and Thomas [5] argued that the second spectrum of frequencies is the result of coupling between pure shear and simple shear modes and concluded that the second frequency spectrum exists only for the special case of hinge-hinge beam. Bhashyam and Prathap [6] exclusively studied the second frequency spectrum of Timoshenko beams using the Finite Element Method (FEM). The study confirmed that the second frequency spectrum exist for Timoshenko beams with any end conditions. Vibration characteristics of Timoshenko beam using FEM was done by Jafarali et al. [7, 8] and it was shown that the second spectrum will not be present in Timoshenko beam vibration model without rotary inertia. The experimental study conducted by Diaz-de-Anda et al. [9] using the electromagnetic acoustic transducer set up, confirmed the existence of second spectrum for Timoshenko beam with free-free boundary condition. Recently, an extensive re-assessment of the Timoshenko beam theory has been published [10].

The wave propagation analysis using the Timoshenko beam theory by Ufuk and Metin [11], obtained two dispersion curves whereas only one curve was observed for Euler-Bernoulli beam theory. Among the different analytical beam models used to find the natural frequency of classical beams, there are nonlinear beam formulations which consider the effect of different parameters more accurately. These nonlinearities were successfully analysed by Hamid et al. [12-15]. Differential Transform Method (DTM) was employed by researchers to solve linear and nonlinear differential equations.

DTM was first proposed by Zhou for the solution of initial value problem in electric circuit analysis. Ho and Chen [16] analysed general elastically end restrained tapered beam using DTM. Free and forced vibration conditions were considered. Chen and Ho [17] obtained the closed form solution of a rotating twisted Timoshenko beam under axial loading with DTM. The effects of the twist angle, spinning speed, and axial force on the natural frequencies of a non-uniform Timoshenko beam were studied. DTM was used to solve Sturm–Liouville eigenvalue problem [18]. Chai and Wang [19] determined the critical buckling load of axially compressed heavy columns with various supports using DTM and the accuracy of results were proved to be very good. DTM was used successfully to solve free vibration of beams resting on elastic foundation [20, 21]. Vibration analysis of pipelines resting on elastic foundations was modelled using Euler Bernoulli and Timoshenko model and solved using DTM by Balkaya et al. [22]. It was observed that the DTM predicted eigenvalues converged to the exact solution and that the rate of convergence is also very good. The eigenvalue analysis on Timoshenko beams were conducted by using various other methods. Free vibration of Timoshenko beam and axisymmetric Mindlin plates based on the pseudospectral method using Chebyshev polynomial was done by Lee and Schultz [23]. This method avoided the process of calculating weighting coefficients and characteristic polynomial.

It is observed that DTM is not been used for reporting second spectrum of Timoshenko beam. The objectives of this paper are to examine the existence of second spectrum of vibration of a Timoshenko beam using DTM and also analyse the mode shapes corresponding to each frequency. The existence of second frequency spectrum has been carefully examined for various end conditions and *L/H* ratios. The end conditions analysed are hinge-hinge, fix-fix, fix-hinge, and fix-free.

2. Analytical Approach to Timoshenko Beam Elastodynamics

The dynamic equation of equilibrium of a Timoshenko beam including the effects of transverse shear and rotary inertia is given by Eqs. (1) and (2).

$$K_s GA \frac{\partial}{\partial x}\left(\psi - \frac{\partial w}{\partial x}\right) + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

$$EI \frac{\partial^2 \psi}{\partial x^2} - K_s GA\left(\psi - \frac{\partial w}{\partial x}\right) - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{2}$$

The general boundary conditions at $x = 0, L$ are:

$$\psi = 0 \tag{3}$$

$$EI \frac{\partial \psi}{\partial x} = 0 \tag{4}$$

$$w = 0 \tag{5}$$

$$K_s GA\left(\psi - \frac{\partial w}{\partial x}\right) = 0 \tag{6}$$

Here ψ is the slope of the deflection curve when the shear deformation is neglected and $(dw/dx - \psi)$ is the shear strain. E is the Young’s modulus, G is the shear modulus of the material of the beam, K_s is the shear correction factor, I and A are the moment of inertia and the sectional area of the beam respectively. L is the length of the beam.

Analytical solution for hinge-hinge beam

For a hinge–hinge case, Eqs. (1) and (2) can be easily coupled (with boundary conditions $w=0$ at $x=0, L$) to the form as in Eq. (7).

$$EI\,\frac{\partial^4 w}{\partial x^4}+\rho A\,\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{K_s G}\right)\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\rho^2 I}{K_s G}\;\frac{\partial^4 w}{\partial t^4}=0 \tag{7}$$

Assuming the system is oscillating at a frequency ω with mode shape $W(x)$, a separation of variables can be assumed in the form $w(x,t)=W(x)\cos\omega t$ for the above equation. Then Eq. (7) is written as,

$$EI\,\frac{\partial^4 w}{\partial x^4}+\rho A\omega^2 W-\rho I\left(1+\frac{E}{K_s G}\right)\omega^2\,\frac{\partial^2 W}{\partial x^2}+\frac{\rho^2 I}{K_s G}\;\omega^4 W=0 \tag{8}$$

A solution to this differential equation, shows two distinct spectra for the natural frequency of the structure, interpreted as, a basic flexural (bending dominated) spectra, and a shear dominated spectra.

For a hinge-hinge beam, the trigonometric function $W(x) = A_n \sin(n\pi x/L)$, satisfies the boundary conditions ($w=0$ at $x=0, L$); Eq. (8) reduces to a quadratic polynomial equation in ω^2.

$$\frac{\rho^2 L^4}{EK_s G}\,\omega_n^4+\left(\frac{\rho A L^4}{EI}+\frac{\rho L^2}{E}\left(1+\frac{E}{K_s G}\right)(n\pi)^2\right)\omega_n^2-(n\pi)^4=0 \tag{9}$$

Here n is the mode number and A_n is the amplitude for n^{th} mode. A solution to Eq. (9) shows two spectra of frequencies [1, 5, 6, 8]. In the present work, DTM is employed and analysed for the existence of second frequency spectra for Timoshenko beam elements with hinge-hinge and other boundary conditions.

3. Differential Transform Method

Differential Transformation Method is based on the Taylor series expansion and, is a transformation technique to obtain analytical solutions of the ordinary and partial differential equations. In this method, the governing differential equation and boundary conditions are transformed into a set of algebraic equations according to certain transformation rules and the solution of these set of equations gives the required solution. Thus, DTM provides an iterative procedure to obtain higher order series in contrast with the Taylor series method where calculation of higher derivatives becomes difficult.

Consider the function $f(x)$ which is analytic in domain D. Let $x=x_0$ represent any point within domain D, the differential transform of $f(x)$ is given by Eq. (10).

$$F[k]=\frac{1}{k!}\left(\frac{d^k f}{dx^k}\right)_{x=x_o} \tag{10}$$

where $f(x)$ is the original function and $F[k]$ is the differential transform. The function $f(x)$ can be defined by the inverse differential transform in Eq. (11).

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d^k f}{dx^k}\right)_{x=x_o}(x-x_o)^k \tag{11}$$

$$f(x)=\sum_{k=0}^{\infty}F[k]\,(x-x_o)^k \tag{12}$$

For practical problems, $f(x)$ is represented by a finite series as given in Eq. (13).

$$f(x)=\sum_{k=0}^{N}F[k]\,(x-x_o)^k \tag{13}$$

which implies that $f(x) = \sum_{k=N+1}^{\infty} F[k]\,(x - x_o)^k$ is negligibly small. N is decided on the convergence of the eigenvalues. Fundamental theorems of differential transform are listed in [11].

3.1. DTM formulation of Timoshenko Beam

Assuming a periodic solution for the displacement and rotation, Eqs. (1) and (2) can be transformed into the eigenvalue problem as given in Eqs. (14) and (15) with w and ψ replaced by $\boldsymbol{W}$ and **S** respectively.

$$\frac{d}{dx}\left[GAK_s\left(S - \frac{dW}{dx}\right)\right] - \omega^2 \rho A W = 0 \tag{14}$$

$$\frac{d}{dx}\left(EI\frac{dS}{dx}\right) - GAK_s\left(S - \frac{dW}{dx}\right) + \omega^2 \rho I S = 0 \tag{15}$$

Introducing the dimensionless parameters in Eq. (16), the Eqs. (14) and (15) can be rewritten as Eqs. (17) and (18).

$$W = \overline{W}\,L\,;\; x = \xi L\,;\; \frac{K_s G}{E} = Q\,;\; \sqrt{\frac{AL^2}{I}} = r\,;\; Qr^2 = \beta\,;\; \Omega = \left(\frac{\rho A \omega^2 L^4}{EI}\right)^{1/4} \tag{16}$$

$$\beta\left(-\frac{d^2\overline{W}}{d\xi^2} + \frac{dS}{d\xi}\right) - \Omega^4 \overline{W} = 0 \tag{17}$$

$$r^2\frac{d^2S}{d\xi^2} + \beta r^2\frac{d\overline{W}}{d\xi} + (-\beta r^2 + \Omega^4)S = 0 \tag{18}$$

Non-dimensional boundary conditions considered for the analysis for a hinge-hinge, fix-fix, fix-hinge and fix-free are represented in Eqs. (19) to (26).

Hinge-Hinge

$$\overline{W}(\xi) = 0\,;\; \frac{dS}{d\xi} = 0 \qquad at\;\; \xi = 0 \tag{19}$$

$$\overline{W}(\xi) = 0\,;\; \frac{dS}{d\xi} = 0 \qquad at\;\; \xi = 1 \tag{20}$$

Fix-Fix

$$\overline{W}(\xi) = 0\,;\; S(\xi) = 0 \qquad at\;\; \xi = 0 \tag{21}$$

$$\overline{W}(\xi) = 0\,;\; S(\xi) = 0 \qquad at\;\; \xi = 1 \tag{22}$$

Fix-Hinge

$$\overline{W}(\xi) = 0\,;\; S(\xi) = 0 \qquad at\;\; \xi = 0 \tag{23}$$

$$\overline{W}(\xi) = 0\,;\; \frac{dS}{d\xi} = 0 \qquad at\;\; \xi = 1 \tag{24}$$

Fix-Free

$$\overline{W}(\xi) = 0\,;\; S(\xi) = 0 \qquad at\;\; \xi = 0 \tag{25}$$

$$\frac{dS}{d\xi} = 0\,;\; S - \frac{d\overline{W}}{d\xi} = 0 \qquad at\;\; \xi = 1 \tag{26}$$

3.2. Solution procedure using DTM

Applying the transformation rules of Differential Transform Method [16], Eqs. (17) and (18) can be rewritten as Eqs. (27) and (28) given below.

$$W[k+2] = \frac{\Omega^4 W[k] - \beta(k+1)S[k+1]}{-\beta(k+1)(k+2)} \quad (27)$$

$$S[k+2] = \frac{-\beta r^2(k+1)W[k+1] - (-\beta r^2 + \Omega^4)S[k]}{r^2(k+1)(k+2)} \quad (28)$$

And the boundary conditions in DTM are also transformed which are expressed in the following set of equations, Eqs. (29) to (36).

Hinge-Hinge

$$W[0] = 0; \quad S[1] = 0 \qquad at\ \xi = 0 \quad (29)$$

$$\sum_{k=0}^{\infty} W[k] = 0; \quad \sum_{k=0}^{\infty} kS[k] = 0 \qquad at\ \xi = 1 \quad (30)$$

Fix-Fix

$$W[0] = 0; \quad S[0] = 0 \qquad at\ \xi = 0 \quad (31)$$

$$\sum_{k=0}^{\infty} W[k] = 0; \quad \sum_{k=0}^{\infty} S[k] = 0 \qquad at\ \xi = 1 \quad (32)$$

Fix-Hinge

$$W[0] = 0; \quad S[0] = 0 \qquad at\ \xi = 0 \quad (33)$$

$$\sum_{k=0}^{\infty} W[k] = 0; \quad \sum_{k=0}^{\infty} k\, S[k] = 0 \qquad at\ \xi = 1 \quad (34)$$

Fix-Free

$$W[0] = 0; \quad S[0] = 0 \qquad at\ \xi = 0 \quad (35)$$

$$\sum_{k=0}^{\infty} S[k] - \sum_{k=0}^{\infty} k\, W[k] = 0; \quad \sum_{k=0}^{\infty} k\, S[k] = 0 \quad at\ \xi = 1 \quad (36)$$

For the purpose of demonstration, the methodology of finding the eigenvalues using DTM is shown for a hinge-hinge end condition. Applying the first set of boundary conditions corresponding to $\xi = 0$, Eqs. (29) and (30) implies $W[0] = 0$, $S[1] = 0$. By substituting, $W[1] = D_1$, $S[0] = D_2$ where D_1 and D_2 are unknown constants and from Eqs. (27) and (28) for k = 0,1,2, 3…the subsequent values, of $W[2]$, $S[2]$, $W[3]$, $S[3]$ ... etc. can be determined in terms of D_1, D_2, and Ω. Substituting all $W[k]$ and $S[k]$ into the second set of boundary conditions corresponding to $\xi=1$, results in two simultaneous equations in Ω corresponding to the N^{th} term. These equations can be rearranged to the matrix form as expressed in Eq. (37).

$$[A][D] = [0] \quad (37)$$

where,

$$[A] = \begin{bmatrix} a_{11}(\Omega) & a_{12}(\Omega) \\ a_{21}(\Omega) & a_{22}(\Omega) \end{bmatrix} \quad (38)$$

$$[D] = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \quad (39)$$

The system of equations in Eq. (37), which is linear in D_1 and D_2 will have a non-trivial solution only if determinant of $\boldsymbol{A}$ is zero, which reduces to the condition as in Eq. (40).

$$\begin{vmatrix} a_{11}(\Omega) & a_{12}(\Omega) \\ a_{21}(\Omega) & a_{22}(\Omega) \end{vmatrix} = 0 \quad (40)$$

The Ω thus obtained after solving Eq. (40) is the non-dimensional frequency of the beam. Therefore $\Omega = \Omega_N^j$ corresponds to the frequency of j^{th} mode. The value of N is decided based on the desirable accuracy required.

$\left|\Omega_j^N - \Omega_j^{N-1}\right| \leq \epsilon$, where ε is a small value taken as 0.001.

4. Results and Discussions

Free vibration analysis of Timoshenko beam with different end conditions are computed using DTM. Analysis is carried out with length to depth (*L/H*) ratios 2, 3, 5, 10 and 20. For all these cases, the parameters used in the computations are K_s = 5/6, E = 1, ρ =1, A = 1 and the Poisson's ratio = 0.3. The set of polynomials derived from the transformations are solved for the non-dimensional frequencies using Mathematica.

FEM is a widely used numerical method for the vibration analysis of Euler Bernoulli and Timoshenko beams. Finite element analysis is also done using a two noded linear Timoshenko beam element with two degrees of freedom (transverse displacement w and rotation ψ) at each node using MATLAB. Shear locking is eliminated using reduced integration [24]. The flexural and shear spectrum frequencies obtained using the DTM are compared with those obtained by finite element analysis and with exact solutions which is available for a hinge-hinge case. The bending spectra frequencies are also compared with the values obtained by pseudo spectral method for different boundary conditions and *L/H* ratios [23].

4.1. Non-dimensional frequencies of hinge-hinge beam

The results of the free vibration analysis of Timoshenko beam with hinge-hinge boundary condition and having *L/H* ratios 2, 3, 5, 10 and 20 are listed in Table 1. The results clearly show two sets of frequencies, the first set is corresponding to the flexural spectrum (bending spectrum) and the second set is corresponding to the shear spectrum.

However, for a thin beam condition where *L/H* = 20, the second set of frequencies were not reflected fully in the DTM solution (in mode3 and mode4) whereas, when *L/H* is 2, 3, 5 and 10, which refers to thick beams, second spectrum of frequencies are clearly evident. This is attributed to the dominant effect of shear in thick beams. The analysis using DTM also identified the pure shear frequency (Mode Zero) where the vibration of the beam is completely in shear mode and no flexure occurs. The pure shear spectra frequency values corresponding to mode zero are observed to be converged fast for all *L/H* values.

Table 1. Non-dimensional frequency for a hinge-hinge Timoshenko beam.

Method	Pure Shear	Mode 1 Spectrum 1st	Mode 1 Spectrum 2nd	Mode 2 Spectrum 1st	Mode 2 Spectrum 2nd	Mode 3 Spectrum 1st	Mode 3 Spectrum 2nd	Mode 4 Spectrum 1st	Mode 4 Spectrum 2nd
				***L/H*=2**					
DTM	5.21	2.72	6.01	4.49	7.3	5.77	8.52	6.80	9.63
Exact	5.21	2.72	6.02	4.49	7.3	5.77	8.52	6.8	9.63
FEM	5.21	2.72	6.02	4.49	7.3	5.77	8.52	6.80	9.64
				***L/H*=3**					
DTM	7.82	2.91	8.44	5.09	9.66	6.73	10.94	8.06	12.18
Exact	7.82	2.91	8.44	5.09	9.66	6.73	10.95	8.07	12.18
FEM	7.82	2.91	8.44	5.09	9.66	6.74	10.95	8.07	12.18
				***L/H*=5**					
DTM	13.03	3.05	13.44	5.67	14.43	7.84	15.66	9.66	16.96
Exact	13.03	3.05	13.44	5.67	14.44	7.83	15.67	9.66	16.96
FEM	13.03	3.05	13.44	5.67	14.44	7.84	15.67	9.66	16.96
[23]		3.05		5.67				9.66	
				***L/H*=10**					
DTM	26.05	3.12	26.27	6.09	26.88	8.84	27.6	11.34	28.86
Exact	26.07	3.12	26.28	6.09	26.89	8.84	27.79	11.34	28.88
FEM	26.07	3.12	26.28	6.09	26.89	8.85	27.78	11.35	28.87
[23]		3.12		6.09		8.84		11.34	
				***L/H*=20**					
DTM	52.11	3.14	52.22	6.23		9.26		12.18	
Exact	52.13	3.14	52.24	6.23	52.56	9.26	53.08	12.18	53.78
FEM	52.13	3.14	52.24	6.23	52.56	9.26	53.08	12.19	53.76
[23]		3.14		6.23		9.26		12.18	

Table 2. Non-dimensional frequency for fix-fix condition.

Method	Mode-1	Mode-2	Mode-3	Mode-4
		***L/H*=2**		
DTM	3.277	4.58	5.782	5.995
FEM	3.278	4.581	5.785	5.996
		***L/H*=3**		
DTM	3.762	5.419	6.862	8.025
FEM	3.763	5.421	6.866	8.031
		***L/H*=5**		
DTM	4.242	6.417	8.283	9.901
FEM	4.243	6.42	8.289	9.909
[23]	4.242	6.418	8.285	9.904
		***L/H*=10**		
DTM	4.579	7.331	9.855	12.144
FEM	4.58	7.334	9.862	12.156
[23]	4.58	7.331	9.856	12.145

Analysis results for other boundary conditions such as fix-fix, fix-hinge, and fix-free conditions for different *L/H* values are presented in Tables 2 to 4 respectively. For these boundary conditions, DTM solution showed only one set of frequency and these are corresponding to the flexural vibrations. This observation is consistent with Abbas and Thomas [5] who suggested that shear spectrum exists only for a Timoshenko beam with hinge-hinge condition. DTM clearly did not identify the shear spectrum for boundary conditions other than hinge-hinge case.

Table 3. Non-dimensional frequency for fix-hinge condition.

Method	Mode-1	Mode-2	Mode-3	Mode-4
		***L/H*=2**		
DTM	3.001	4.548	5.447	5.784
FEM	3.002	4.55	5.449	5.787
		***L/H*=3**		
DTM	3.359	5.273	6.795	7.964
FEM	3.359	5.275	6.798	7.967
		***L/H*=5**		
DTM	3.665	6.072	8.073	9.784
FEM	3.666	6.074	8.078	9.792
[23]	3.666	6.073	8.074	9.786
		***L/H*=10**		
DTM	3.852	6.730	9.365	11.757
FEM	3.852	6.733	9.371	11.768
[23]	3.852	6.731	9.366	11.758

Table 4. Non-dimensional frequency for fix-free condition.

Method	Mode-1	Mode-2	Mode-3	Mode-4
		***L/H*=2**		
DTM	1.729	3.362	4.861	5.539
FEM	1.729	3.363	4.863	5.541
		***L/H*=3**		
DTM	1.802	3.845	5.688	7.045
FEM	1.802	3.846	5.690	7.049
		***L/H*=5**		
DTM	1.847	4.285	6.610	8.517
FEM	1.847	4.286	6.613	8.522
		***L/H*=10**		
DTM	1.868	4.572	7.415	9.986
FEM	1.868	4.573	7.418	9.993

4.2. Convergence analysis

The convergence of the pure shear frequency and frequencies corresponding to the first and second mode of Timoshenko beam with hinge-hinge condition and *L/H=2* is plotted in Fig. 1.

The dimensionless frequency values are plotted against the value of N at which the convergence of eigenvalues occurred in DTM computation. The pure shear mode converged fast. The first mode of the bending spectra is converged next followed by the first mode of shear spectra. The convergence of frequencies in mode 3, mode 4 and mode 5 are shown in Fig. 2. The rate of convergence of shear spectrum frequency is found to be faster in the case of third and fourth mode.

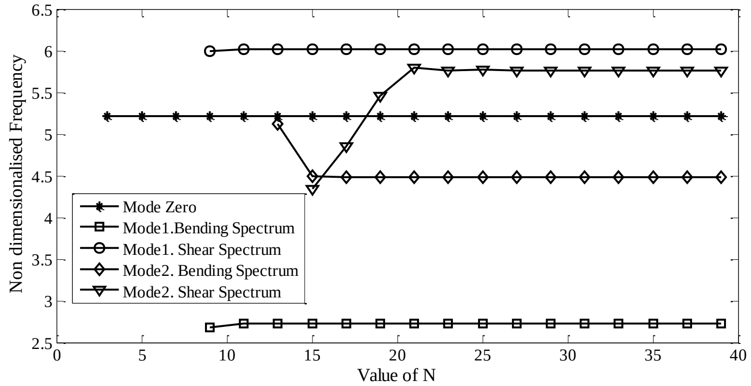

Fig. 1. Convergence plot for dimensionless frequencies of Mode zero, Mode1 and Mode2 for hinge-hinge condition with *L*/*H*=2.

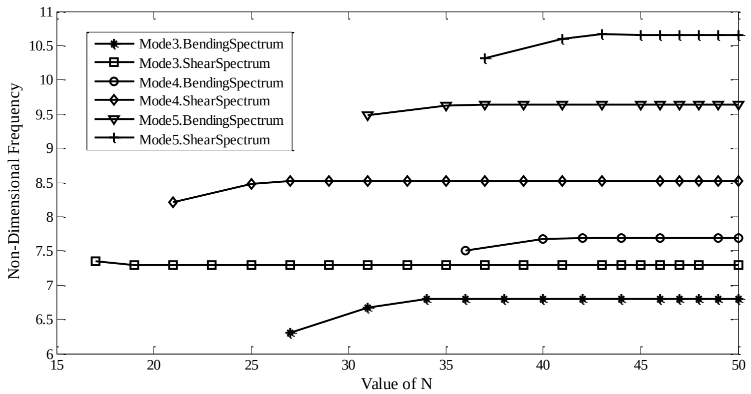

Fig. 2. Convergence plot for dimensionless frequencies for Mode3, Mode4 and Mode5 for hinge-hinge condition with *L/H*=2.

4.3. Comments on eigenfunctions of the two spectra

For the further investigation of the flexural and shear spectrum of Timoshenko beam with hinge-hinge support condition, the mode shapes are plotted using DTM. The mode shapes are extracted from the fundamental definition of the differential transform given by the Eq. (13).

From Table 1, it is observed that Ω_1^1= 2.723 which is the first spectrum dimensionless frequency of the first mode. The convergence of this value occurred with a value of N= 14. Substituting Ω_1^1 into F [*0*], F [*1*],, F [*14*] and using Eq. (13), we obtain the closed form series solution of the first spectrum of the first mode shape. Similarly, the mode shapes corresponding to each frequency are plotted. The mode shapes corresponding to bending spectra and shear spectra are presented for hinge-hinge case with *L/H*=2 in Figs. 3 to 6.

It is observed that in the investigation of eigenfunction of the Timoshenko beam, there are two components, the displacement and rotation. The normalized eigenfunctions show that the displacement component corresponding to the frequency of a particular mode in bending spectra and shear spectra are similar.

The rotation components are also found to follow the same mode shape in both spectra. However, they differ in the relative values of the displacement and rotation component in both the spectra. In the pure shear mode, as shown in Fig. 3, the section rotates without any bending. In this mode, it can be seen that the displacement component is zero.

The first mode for the bending and shear spectra are presented in Fig. 4. The mode shapes for the displacement components of the first spectrum (bending) and the displacement component of second spectrum(Shear) are found to be similar with different amplitude. Similar trend is seen for rotation components also. However, in the case of bending spectra, the ratio of displacement component to rotation component is more compared to the corresponding the values from the eigenfunction of the shear spectra frequency. Same observations could be made for other *L*/*H* ratios where the bending is less in shear spectra modes compared to the bending in the corresponding bending spectra modes. Second and third modes of the shear and bending spectra are presented in Figs. 5 and 6. As mentioned before, the shear is dominated in the shear spectra modes compared to the corresponding bending spectra modes.

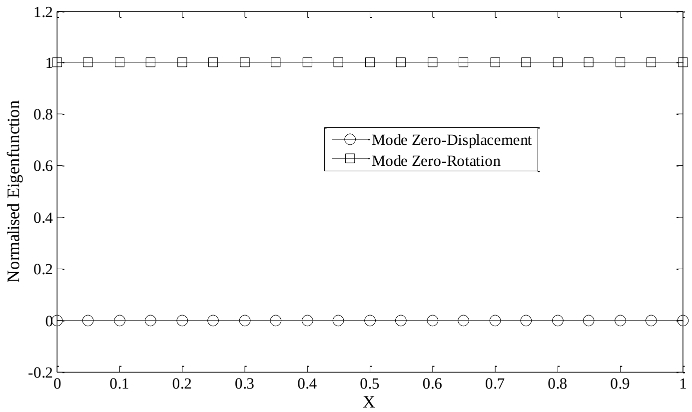

Fig. 3. Displacement and rotation components for hinge-hinge condition (*L/H*=2) for pure shear mode (mode zero).

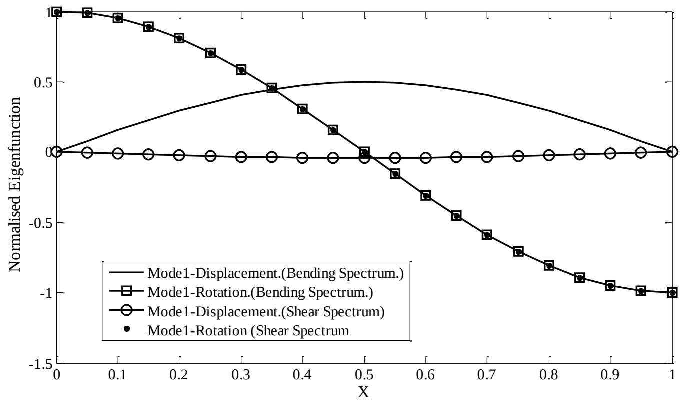

Fig. 4. Displacement and rotation modes of hinge-hinge condition (*L/H*=2) for Mode1 bending spectra and shear spectra.

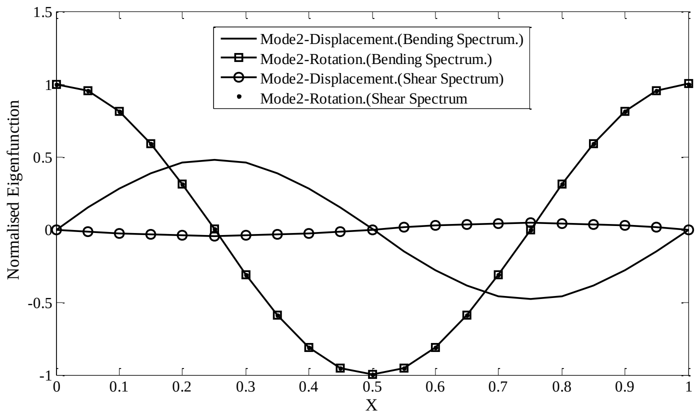

Fig. 5. Displacement and rotation modes of hinge-hinge condition (*L/H*=2) for Mode2 bending spectra and shear spectra.

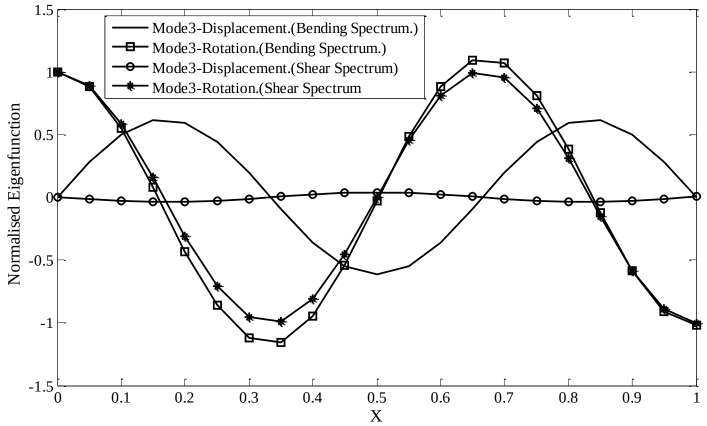

Fig. 6. Displacement and rotation modes of hinge-hinge condition (*L/H*=2) for Mode3 bending spectra and shear spectra.

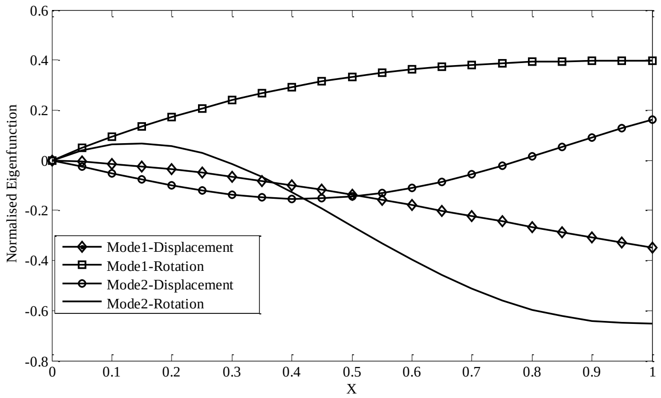

Fig. 7. Displacement and rotation modes of cantilever beam (*L/H*=2) for first and second modes.

The mode shapes of the beam with fix-free boundary conditions are also presented in Fig. 7 for *L*/*H*=2 where only the flexural mode of vibration is visible. The analysis is done with other support conditions like fix-hinge and fix-fix also, where the second spectra were not visible. Hence the second spectra are visible only for a Timoshenko beam with hinge-hinge support condition.

5. Conclusions

The analytical existence of two spectra of frequency for a Timoshenko Beam is due to the factorization of the transcendental frequency equation for hinge-hinge condition [5]. For the case of hinge-hinge case, the strong coupling between the effects of shear and the rotation of the cross section leads to two family of frequency curves. Whereas in the case of fix-free, fix-fix and fix-hinge, even though the coupling exists, there is no discontinuity in the variation of frequency parameter with the rotary inertia parameter and hence there is no separation of frequency spectra. From the results of DTM analysis of Timoshenko beam, it seen that the shear and rotary inertia effects are of great importance in the case of Timoshenko beam. This fact is evident from the mode shapes plotted as in Figs. 4 to 6, where the relative magnitude of the rotation component in the second spectrum is found to be predominant when compared to the bending spectrum.

In the present study, free vibration of Timoshenko beam was examined using DTM. The eigenvalues and corresponding eigenfunctions were derived for various end conditions using the DTM. From the analysis it is evident that, DTM can capture the pure shear spectrum and two different frequency spectrum which arises in the case of a Timoshenko beam with hinge-hinge condition. Some concluding observations include:

- Second spectrum of frequency is evident only for a Timoshenko beam with hinge-hinge end condition.
- In pure shear spectrum, the bending component is absolutely zero.
- The first set of frequency in each mode is attributed to the bending spectrum and the second set of frequency is attributed to the shear spectrum.
- The two separate frequency sets are clearly evident in the frequency results as well as the mode shapes derived using the present method. The mode shapes are coupled with displacement and rotation components and the mode shapes of the corresponding frequencies of these two spectra are similar. They differ in the ratio of the rotation and displacement components in the respective eigenfunction set.
- The existence of the second frequency set of values is clear only for small *L/H* ratios even for hinge-hinge case. For *L/H* ratio of 2, 3, 5 and 10 the second set of values were clearly determined.
- As the *L/H* ratio is increased, in which case the beam is more idealized as a Euler Bernoulli beam, the second spectra disappears gradually. Hence it is seen that the second spectra which is dominated by shear, is characteristic to Timoshenko beam.
- Existence of second spectrum was not observed in other end conditions like fix-fix, fix-hinge and fix-free.

The accuracy of results derived by the DTM shows a very good agreement with the analytical and numerical results.

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