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APPROXIMATE OPTIMAL CONTROL OF LINEAR TIME-DELAY SYSTEMS VIA HAAR WAVELETS

AKBAR H. BORZABADI*, SOLAYMAN ASADI

School of Mathematics and Computer Science, Damghan University, Damghan, Iran
*Corresponding Author: borzabadi@du.ac.ir

Abstract

In this paper, Haar wavelet benefits are applied to the optimal control of linear time-delay systems. A discretized form of optimal control problem at collocation points based on some useful properties of Haar wavelets transforms original problem into a nonlinear programming (NLP). The given numerical examples show the accuracy of the presented scheme in comparison with some other methods.

Keywords: Haar wavelet, Optimal control problem, Discretization, Linear time-delay system, Nonlinear programming.

1. Introduction

Over the last few years we have witnessed an ever increasing interest in the study of control processes governed by different systems. One of the most important of these systems are delay systems. Time-delay often appears in many control systems (such as aircraft, chemical or process control systems) either in the state, the control input, or the measurements. Due to presenting delay and its important consideration, in many practical systems [1, 2], control of time-delay systems has been interested by many engineers and scientist. Since the analytical methods, especially in optimal control of time-delay systems, have less ability to implement, the different numerical methods to overcome the problems of exact methods have been devised. Some of these techniques include, iterative dynamic programming [3], steepest descent based algorithm [4], Chebyshev series [5], Laguerre polynomials [6], Block-pulse functions [7], Hybrid of block-pulse and Legendre polynomials [8], Legendre multiwavelets [9], Walsh functions [10].

Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful

Nomenclatures

a_i	Haar coefficient
$H_M(t)$	Haar matrix
P	Operational integration matrix
$D(\tau)$	Delay operational matrix
$0_{(M/2)\times M/2)}$	Null matrix of order $(M/2)\times(M/2)$

Greek Symbols

ε	Integral square error
$\phi_i(t)$	A group of square waves
$\phi(t-\tau)$	Delay function of $\varphi(t)$

Abbreviations

NLP	Nonlinear programming

mathematical tool. Haar wavelets have the simplest orthogonal series with compact support. In characteristics makes Haar wavelets good candidate for application to optimal control problems [11]. The collocation methods developed to solve optimal control problems generally fall into two categories, local collocation [12] and global orthogonal collocation [13]. In local collocation methods, the time interval considered is divided into a series of subintervals within which the integration rule must be satisfied. In recent years, more attention has been focused on global orthogonal collocation methods such as Chebyshev, Legendre and some other. By expanding the state and control variables into piecewise-continuous combination of these interpolating polynomials and derivatives, then, the objective function and system constraints are all converted into algebraic equations with unknown coefficients.

In this paper, we introduce an alternative method to solve the linear optimal control with delay systems. We introduce the Haar wavelets theory and properties including the Haar wavelets basis and its integral operational matrix [11]. The delay and product are given. Then we will assume that the control variables and derivative of the state variables in the optimal control problems may be expressed in the form of Haar wavelets and unknown coefficients. By using the Haar operational integration matrix we find $X(t)$. The delay vector $X(t-\tau_1)$ and $U(t-\tau_2)$ can be calculated by using the delay operational matrix and Haar operational integration matrix. Therefore, all variables in the time-delay system are expressed as series of the Haar family and operational matrix and delay operational matrix. Finally, the task of finding the unknown parameters that optimize the designate performance while satisfying all constraints is performed by a nonlinear programming solver.

In this paper, first Haar wavelets and its properties is introduced. Then the approximation of a function by Haar wavelets is discussed. By introducing operational integration matrix and delay operational matrix, Haar discretization method is established. After, describing the formulation of the optimal control problem with delays, the proposed method is used in the analysis of linear time-delay systems. Finally, by some numerical examples the proficiency of the given approach is examined and its results compared with other methods.

2. Haar Wavelets and Its Properties

The orthogonal set of Haar wavelets $\phi_i(t)$ is a group of square waves with magnitude +1 or -1 in some intervals and zeros elsewhere.

$$\phi_0(t) = 1, 0 \le t < 1, \tag{1}$$

$$\phi_1(t) = \begin{cases} 1 & if \ \ 0 \le t < \frac{1}{2}, \\ -1 & if \ \ \frac{1}{2} \le t < 1, \end{cases} \tag{2}$$

$$\varphi_i(t) = \varphi_1(2^j t - k) = \begin{cases} 1 & \text{if } \ \frac{k}{2^j} \le t < k2^j + \frac{1}{2^{j+1}}, \\ -1 & \text{if } \ \frac{k}{2^j} + \frac{1}{2^{j+1}} \le t < \frac{k}{2^j} + \frac{1}{2^j}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

for $i = 2^j + k, j = 0, \cdots, M, k = 0, \cdots, 2^j - 1,$ integer $m = 2^j, (j = 0,1,\cdots,J)$, indicates the level of the wavelet and $k = 0,1,\cdots,M-1$ is the translation parameter. Maximal level of resolution is J . The maximal value of i is $M = 2^{J+1}$. A simple calculation shows that

$$\int_0^1 \varphi_i(t)\varphi_l(t) = \begin{cases} \frac{1}{M} & if \ \ i = l = 2^j + k, \\ 0 & if \ \ i \ne l. \end{cases}$$

(4)

Consequently, the functions $\varphi_i(t)$ are orthogonal. This allows us to transform any function square integrable on the interval time [0,1] into Haar wavelets series.

3. Function Approximation by Haar Wavelets

We just pointed out that a square integrable function can be expressed in terms of Haar orthogonal basis on interval $\tau \in [0,1]$. However, before the procession to this transfer, it is necessary to unify the time interval. By using a linear transformation, the actual time t can be expressed as a function of τ via $t = [(t_f - t_0)\tau + t_0]$ where t_0 is the initial time and t_f is the final time in a square integrable function $f(t)$.

Any function $f(t)$ which is square integrable in the interval [0,1] can be expanded in a Haar series with an infinite number of terms.

$$f(t) = \sum_{i=0}^{\infty} a_i \varphi_i(t), i = 2^j + k, j \ge 0, 0 \le k < 2^j, t \in [0,1), \tag{5}$$

where the Haar coefficients

$$a_i = 2^j \int_0^1 f(t)\varphi_i(t)dt, \tag{6}$$

are determined in such a way that the integral square error

$$\varepsilon = \int_0^1 (f(t) - \sum_{i=1}^{M-1} a_i \varphi_i(t))^2 dt, \tag{7}$$

is minimum. Here ε is vanished when M tends to infinity. Usually, the series expansion of (5) contains an infinite number of terms for smooth $f(t)$. If $f(t)$ is a piece wise constant or may be approximated as a piecewise constant, then the summation (5) will be terminated after M terms, that is,

$$F(t) \approx \sum_{i=0}^{M-1} a_i \varphi_i(t) = A^T \Psi_M(t), \tag{8}$$

where the coefficient vector $A = [a_0, a_1, ..., a_{M-1}]^T$ and $\Psi_M(t) = [\varphi_0, \varphi_1, ..., \varphi_{M-1}]^T$.

Let us define the collocation points $t_s = (s - 0.5)/M$, $(s = 1, ...M)$. With these chosen collocation points, the function is discretized into a series of nodes with equivalent distances. Let the Haar matrix H be the combination of $\Psi_M(t)$ at all the collocation points. Thus,

$$H_M(t) = [h_1, ..., h_M]$$

$$= [\Psi_M(t_0), ..., \Psi_M(t_{M-1})] = \begin{bmatrix} \varphi_0(t_0) & \varphi_0(t_1) & \cdots & \varphi_0(t_{M-1}) \\ \varphi_1(t_0) & \varphi_1(t_1) & \cdots & \varphi_1(t_{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{M-1}(t_0) & \varphi_{M-1}(t_1) & \cdots & \varphi_{M-1}(t_{M-1}) \end{bmatrix}_{M \times M}. \tag{9}$$

For example,

$$H_2 = [\Psi_2(t_0) \quad \Psi_2(t_1)] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Therefore, the function $f(t)$ may be approximated as

$$f(t_s) = c_{1\times M}^T H_{M\times M}. \tag{10}$$

4. Integration of Haar Wavelets

In the wavelet analysis for a dynamic system, all functions need to be transformed into Haar series. Since the differentiation of Haar wavelets always results in impulse functions which should be avoided, the integration of Haar wavelets is preferred, which should be expandable into Haar series with Haar coefficient matrix P

$$\int_0^t \Psi(t')dt' = P\Psi(t), \tag{11}$$

where P is the $M \times M$ operational integration matrix which satisfies the following recursive formula,

$$P_M = \begin{bmatrix} P_{M/2} & \frac{-1}{2M} H_{M/2} \\ \frac{1}{2M} H_{M/2}^{-1} & 0 \end{bmatrix}, P_1 = [\frac{1}{2}], \tag{12}$$

which is given in [14] and $0_{(M/2)\times(M/2)}$ is a null matrix of order $(M/2)\times(M/2)$.

5. Delay Operational Matrix of Haar Wavelets

The delay function $\varphi(t-\tau)$ is the shift of $\varphi(t)$ defined in (3) along the time axis by τ. The delay operational matrix $D(\tau)$ is given by

$$\varphi(t-\tau)\approx D(\tau)\varphi(t), t>\tau, 0\le t<1, \tag{13}$$

$\tau\in[0,1]$ is the delay parameter. First, we find the matrix $D(\tau)=[d_{ij}]$ for $0\le\tau\le\frac{1}{2}$.

The four basis functions are given by $\varphi_0,\varphi_1,\varphi_2,\varphi_3$. By (8) and (13) we have

$$\begin{bmatrix}\varphi_0(t-\tau)\\ \varphi_1(t-\tau)\\ \varphi_3(t-\tau)\\ \varphi_4(t-\tau)\end{bmatrix}=[d_{ij}(\tau)]\begin{bmatrix}\varphi_0(t)\\ \varphi_1(t)\\ \varphi_3(t)\\ \varphi_4(t)\end{bmatrix}, i,j=1,2,3,4,$$

where $\varphi_0(t-\tau)=1$, $\tau\le t<1$, and

$$\varphi_1(t-\tau)=\begin{cases}1 & \text{if } \tau\le t<\frac{1}{2}+\tau,\\ -1 & \text{if } \frac{1}{2}+\tau\le t<1,\end{cases}$$

$$\varphi_2(t-\tau)=\begin{cases}1 & \text{if } \tau\le t<\frac{1}{4}+\tau,\\ -1 & \text{if } \frac{1}{4}+\tau\le t<\frac{1}{2}+\tau,\end{cases}$$

and

$$\varphi_3(t-\tau)=\begin{cases}1 & \text{if } \frac{1}{2}+\tau\le t<\frac{3}{4}+\tau,\\ -1 & \text{if } \frac{3}{4}+\tau\le t<1.\end{cases}$$

To find the entries $d_{ij}(\tau)$, $i,j=1,2,3,4$, we use the inner product. For example if $\tau=0.1$, we have $d_{11}=<\varphi_0(t-\tau),\varphi_0(t)>=\int_0^1\varphi_0(t-\tau)\varphi_0(t)dt=0.9$, $d_{31}=<\varphi_3(t-\tau),\varphi_0(t)>=\int_0^1\varphi_3(t-\tau)\varphi_0(t)dt=0$. If we calculate all $d_{ij}(\tau)$ as d_{11} and d_{31} the 4×4 operational matrix $D(\tau)$ is obtained. In particular, we have

$$D_{4\times4}(0.1)=\begin{bmatrix}0.9 & -0.1 & -0.1 & 0\\ 0.1 & 0.7 & -0.1 & 0.2\\ 0 & 0.2 & 0.2 & -0.1\\ 0.1 & -0.1 & 0 & 0.2\end{bmatrix}.$$

In a similar manner, if we use the vector function $\varphi(t)$ with dimension $2^{n+1}\times1$, then $2^{n+1}\times2^{n+1}$ delay matrix $D(\tau)$ with $0\le\tau\le12^n$ can be obtained as follows,

$$d_{il}=\int_0^1\varphi_i(t)\varphi_l(t)dt=\begin{cases}2^{-j} & \text{if } i=l=2^j+k,\\ 0 & \text{if } i\ne l.\end{cases}$$

Note that for any dimension if $\tau=0$ then matrix is diagonal.

6. Problem Statement

Consider a linear system with delays in both the state and control described by

$\dot{X}(t) = A(t)X(t) + B(t)X(t-\tau_1) + E(t)U(t) + S(t)U(t-\tau_2),$ (14)

with initial data

$X(0) = X_0,$ (15)

$X(t) = \phi(t), t \in [-\tau_1, 0],$ (16)

$U(t) = \theta(t), t \in [-\tau_2, 0],$ (17)

where X is an m-vector of state; U is an q-vector of input; $A(t)$, $B(t)$, $E(t)$ and $S(t)$ are continuous matrix functions of the time of appropriate dimensions, X_0 is a constant specified vector. τ_1 and τ_2 are delays in state and control, respectively, and the initial function $\phi(t)$ and $\theta(t)$ are continuous in their respective intervals.

The problem is to minimize

$J = H(t_f, X_f) + \int_0^{t_f} L(t, X, U)dt,$ (18)

where H is a scalar function of the final time t_f and final state variables and $L(t, X, U)$ is a scalar function of the time, state X and control U.

7. Haar Discretization and Time-Delay Systems Analysis

We discretize the functions $\varphi_i(t)$ by dividing the interval $\tau \in [0,1]$, to M parts of equal length $\Delta t = 1/M$ and introduce the collocation points $\tau_k = (k - 0.5)/M, k = 1,...,M$, where M is the number of nodes used in the discretization and also is the maximum wavelet index number.

We approximate state variables $\dot{x}(\tau)$ and control variables $u(\tau)$ by Haar wavelets with M collocation points, i.e.,

$\dot{x}(\tau) \approx c_x^T \Psi_M(\tau),$ (19)

$u(\tau) \approx c_u^T \Psi_M(\tau),$ (20)

where $c_x^T = [c_{x_1}, ..., c_{x_M}]$ and $c_u^T = [c_{u_1}, ..., c_{u_M}]$. Using the operational integration matrix P defined in (11)

$x(\tau) = \int_0^{\tau} \dot{x}(\tau')d\tau' + x_0 = \int_0^{\tau} c_x^T \Psi_M(\tau')d\tau' + x_0 = c_x^T P \Psi_M(\tau) + x_0.$ (21)

As stated in (11), the expansion of the matrix $\Psi_M(\tau)$ at the M collocation points will the yield the $M \times M$ Haar matrix $H = [h_1, ..., h_M]$ it fellows that

$\dot{x}(\tau_k) = c_x^T h_k, u(\tau_k) = c_u^T h_k, x(\tau_k) = C_x^T p h_k + x_0, k = 1,...,M.$ (22)

Now we focus on the analysis of time-delay systems. First choose N_k as following manner,

$N_k = \lfloor M\tau_k + 0.5 \rfloor,$ (23)

$k = 1, 2$ and let $N_1 \le N_2$. Let

$X(\tau) = [x_1(\tau), x_2(\tau), ..., x_m(\tau)],$ (24)

$U(\tau) = [u_1(\tau), u_2(\tau), ..., u_q(\tau)],$ (25)

$\hat{\Psi}_M(\tau) = I_m \otimes \Psi_M(\tau),$ (26)

$\bar{\Psi}_M(\tau) = I_q \otimes \Psi_M(\tau),$ (27)

where I_m and I_q dimensional identity matrices and $\otimes$ denotes kroneker product [15]. By (19)-(22) each of $x_i(t), i=1,2,...,m$ and $u_j(t), j=1,2,...,q$ can be written as

$$x_i(\tau)=c_{x_i}^T P\Psi_M(\tau)+x_i(0), \tag{28}$$

$$u_j(\tau)=c_{u_j}^T \Psi_M(\tau), \tag{29}$$

where $c_{x_i}=[c_{x_{i1}},c_{x_{i2}},...,c_{x_{iM}}]^T$ and $c_{u_j}=[c_{u_{j1}},c_{u_{j2}},...,c_{u_{jM}}]^T$. Using (23)-(29)

$$\dot{X}(\tau)=C_X^T\hat{\Psi}_M(\tau),\ X(\tau)=C_X^T\hat{P}\hat{\Psi}_M(\tau)+X(0), \tag{30}$$

and also

$$U(\tau)=C_U^T\bar{\Psi}_M(\tau), \tag{31}$$

where $C_X=[c_{x_1},c_{x_2},...,c_{x_m}]^T$, $C_U=[c_{u_1},c_{u_2},...,c_{u_q}]^T$ and $\hat{P}=I_m\otimes P$. By discussion in Sec.3 and a similar notation we can write

$$\phi(\tau-\tau_1)=C_\phi^T\hat{\Psi}_M(\tau), \tag{32}$$

$$\theta(\tau-\tau_2)=C_\theta^T\bar{\Psi}_M(\tau). \tag{33}$$

Using (13), (30) and (31), it can be concluded that

$$X(\tau-\tau_1)=\begin{cases}\phi(\tau-\tau_1) & if\ \ 0\le\tau<\tau_1,\\ C_X^T\hat{P}\hat{D}(\tau_1)\hat{\Psi}_M(\tau)+X(0) & if\ \ \tau\ \tau_1\le\tau<t_f,\end{cases} \tag{34}$$

$$U(\tau-\tau_2)=\begin{cases}\theta(\tau-\tau_2) & if\ \ 0\le\tau<\tau_2,\\ C_U^T\bar{D}(\tau_2)\bar{\Psi}_M(\tau) & if\ \ \tau_2\le\tau<t_f,\end{cases} \tag{35}$$

where $\hat{D}=I_m\otimes D$ and $\bar{D}=I_q\otimes D$. Similarly each entries of $A(t)$, $B(t)$, $E(t)$ and $S(t)$ may be expanded by (8) and thus

$$A(t)=A\hat{\Psi}_M(t), \tag{36}$$

$$B(t)=B\hat{\Psi}_M(t), \tag{37}$$

$$E(t)=E\bar{\Psi}_M(t), \tag{38}$$

$$S(t)=S\bar{\Psi}_M(t), \tag{39}$$

where

$$A=\begin{bmatrix}c_{a_{11}} & c_{a_{12}} & \cdots & c_{a_{1m}}\\ c_{a_{21}} & c_{a_{22}} & \cdots & c_{a_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ c_{a_{n1}} & c_{a_{n2}} & \cdots & c_{a_{nm}}\end{bmatrix}_{n\times m},\ B=\begin{bmatrix}c_{b_{11}} & c_{b_{12}} & \cdots & c_{b_{1m}}\\ c_{b_{21}} & c_{b_{22}} & \cdots & c_{b_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ c_{b_{n1}} & c_{b_{n2}} & \cdots & c_{b_{nm}}\end{bmatrix}_{n\times m}$$

$$E=\begin{bmatrix}c_{e_{11}} & c_{e_{12}} & \cdots & c_{e_{1q}}\\ c_{e_{21}} & c_{e_{22}} & \cdots & c_{e_{2q}}\\ \vdots & \vdots & \ddots & \vdots\\ c_{e_{n1}} & c_{e_{n2}} & \cdots & c_{e_{nq}}\end{bmatrix}_{n\times q},\ S=\begin{bmatrix}c_{s_{11}} & c_{s_{12}} & \cdots & c_{s_{1q}}\\ c_{s_{21}} & c_{s_{22}} & \cdots & c_{s_{2q}}\\ \vdots & \vdots & \ddots & \vdots\\ c_{s_{n1}} & c_{s_{n2}} & \cdots & c_{s_{nq}}\end{bmatrix}_{n\times q}.$$

When the Haar collocation method is applied in the optimal control problem with time-delay system, the variables can be set as the unknown coefficients vector of the derivative of the state variables and control variables together with initial and final times, that is

$[C_{X_1}, C_{X_2}, ..., C_{X_M}, C_{U_1}, ..., C_{U_M}, t_f, t_0]$.

Consider the cost functional (18) by

$J = H(t_f, X(M)) + (t_f - t_0)\int_0^1 L(\tau, C_X^T \hat{P}\hat{\Psi}_M(\tau) + X_0, C_U^T \bar{\Psi}_M(\tau))dt.$

Since the Haar wavelets are expected to be constant steps at each time interval, the above equation can be simplified as

$J = H(t_f, X(M)) + (t_f - t_0)M\sum_{k=1}^{M} L(\tau_k, C_X^T \hat{P}\hat{\Psi}_M(\tau_k) + X_0, C_U^T \bar{\Psi}_M(\tau_k)),$

By substituting $\dot{X}$, U, and X in (14) and using (23)-(39), for $k = 1, ..., N_1$, we have

$$C_X^T \hat{\Psi}_M(\tau_k) = (t_f - t_0)((A\hat{\Psi}_M(\tau_k))(C_X^T \hat{P}(\tau_1)\hat{\Psi}_M(\tau_k)) + X(0))$$
$$+ B\hat{\Psi}_M(\tau_k)\phi(\tau_k - \tau_1) + (E\bar{\Psi}_M(\tau_k))(C_U^T \bar{\Psi}_M(\tau_k)) + S\bar{\Psi}_M(\tau_k)\theta(\tau_k - \tau_2)),$$

and also for $k = N_1 + 1, ..., N_2$,

$$C_X^T \hat{\Psi}_M(\tau_k) = (t_f - t_0)((A\hat{\Psi}_M(\tau_k))(C_X^T \hat{P}(\tau_1)\hat{\Psi}_M(\tau_k)) + X(0))$$
$$+ B\hat{\Psi}_M(\tau_k)(C_X^T \hat{P}\hat{D}(\tau_1)\hat{\Psi}_M(\tau_k) + X(0)) + (E\bar{\Psi}_M(\tau_k))(C_U^T \bar{\Psi}_M(\tau_k)) + S\bar{\Psi}_M(\tau_k)\theta(\tau_k - \tau_2)).$$

Also for $k = N_2 + 1, ..., M$,

$$C_X^T \hat{\Psi}_M(\tau_k) = (t_f - t_0)((A\hat{\Psi}_M(\tau_k))(C_X^T \hat{P}(\tau_1)\hat{\Psi}_M(\tau_k)) + X(0))$$
$$+ B\hat{\Psi}_M(\tau_k)(C_X^T \hat{P}\hat{D}(\tau_1)\hat{\Psi}_M(\tau_k) + X(0)) + (E\bar{\Psi}_M(\tau_k))(C_U^T \bar{\Psi}_M(\tau_k))$$
$$+ S\bar{\Psi}_M(\tau_k)(C_U^T \bar{D}(\tau_2)\bar{\Psi}_M(\tau_k))).$$

Note that in (9) we pointed that for $k = 1, ..., M$, $\Psi_M(\tau_k) = h_k$.

Since the first and last collocation points are not set as the initial and final time, the initial and final state variables are calculated according to

$$X_0 = X_1 - \dot{X}(1)/2M,$$
$$X_f = X_M + \dot{X}(M)/2M.$$

In this way, the optimal control of time-delay systems transformed into NLP or LP problem.

8. Numerical Results

In this section, The results of applying the method in three numerical examples are presented.

Example 8.1 Consider the following optimal control problem of linear time-delay system

$\dot{x}(t) = 4tx(t) + 2x(t - 1/2) + 2u(t),$

$x(t) = 1, -1/2 \le t \le 0,$

with associated quadratic cost functional to be minimized

$J = \frac{1}{2}\int_0^1 (4x^2(t) + 4u^2(t))dt.$

Using the Haar wavelets collocation method with M =16 collocation point and by (17) and (23)-(39) we have $\tau_1 = 12$,

$c_x^T h_k = 4\tau_k(c_x^T Ph_k + x(0)) + 2 + 2c_u^T h_k, k = 1, 2, ..., N_1,$

$c_x^T h_k = 4\tau_k(c_x^T Ph_k + x(0)) + 2(c_x^T PD(\tau_1)h_k + x(0)) + 2c_u^T h_k, k = N_1 + 1, ..., M,$

$$J = \frac{1}{2M}\sum_{k=1}^{M}(4(c_x^T Ph_k + x(0))^2 + 4(c_u^T h_k)^2),$$

where c_x and c_u are the unknown variables of NLP and by (23) $N_1 = 8$. The obtained minimum value of the cost functional is $J = 4.7325$ which is much better than $J = 5.1713$, reported in [9]. Again the results has been examined using 32 collocation points. In Figs. 1 and 2, one can observe the diagram of approximate optimal control and state functions, respectively.

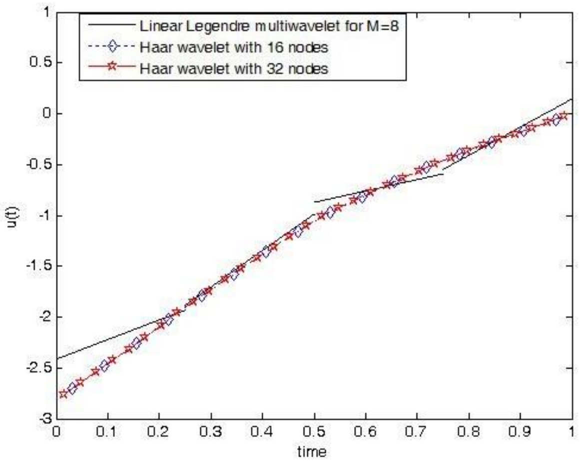

Fig. 1. The approximate optimal control input in Example 8.1.

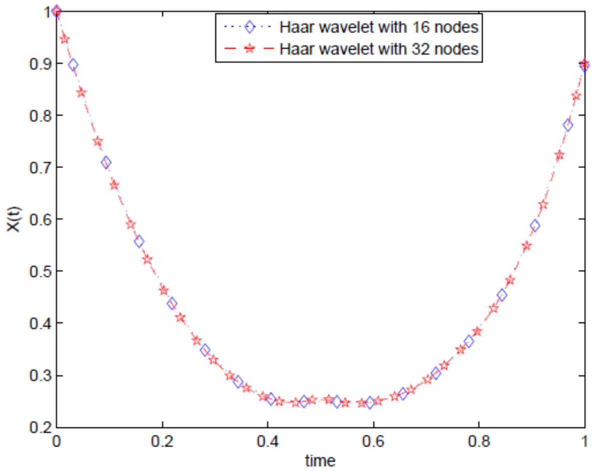

Fig. 2. The approximate optimal trajectory in Example 8.1.

Example 8.2 Consider the problem of minimizing

$$J = 12X^T(1)\begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}X(1) + 12\int_0^1 (2u^2(t))dt, \tag{40}$$

subject to the system of the delayed differential equations

$$\dot{X}(t) = \begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}X(t) + \begin{bmatrix}0 & 0\\ 0 & -2\end{bmatrix}X(t-12) + \begin{bmatrix}0\\ 2\end{bmatrix}u(t), 0 \le t \le 1, \tag{41}$$

$$X(t) = [x_1(t) \quad x_2(t)]^T,\ X(0) = [10 \quad 0]^T,\ X(t) = [0 \quad 0]^T,\ -\frac{1}{2} \le t \le 0. \tag{42}$$

As previous example, the minimization of J subject to (41) and (42) has been obtained using the proposed method. Using 16 collocation points for Haar wavelets discretization method, the optimal value is obtained $J = 2.6986$, which is better than $J = 3.43254$ and $J = 3.3991$, reported in [9] and [16], respectively. The control variable $u(t)$ and the state variables $x_1(t)$, $x_2(t)$ for two different number of collocation points, $M = 16$ and $M = 32$, depicted in Figs. 3 and 4, respectively.

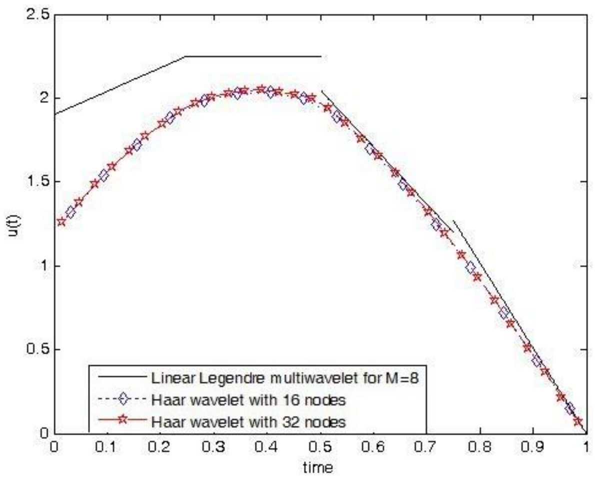

Fig. 3. The approximate optimal control input in Example 8.2.

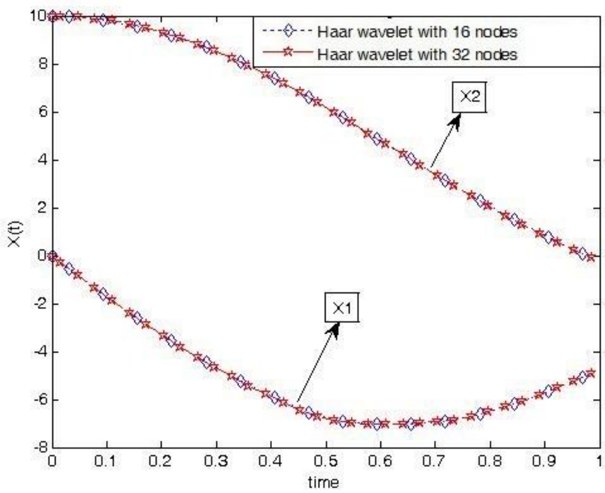

Fig. 4. The approximate optimal trajectory in Example 8.2.

Example 8.3 In this example, the delay is considered in control and state variables. The problem is minimization of the functional

$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,$$

subject to delayed differential equation

$$\dot{x}(t) = -x(t) + x(t - \frac{1}{3}) + u(t) - \frac{1}{2}u(t - \frac{2}{3}),$$

$$x(t) = 1, t \in [-\frac{1}{3}, 0],$$

$$u(t) = 1, t \in [-\frac{2}{3}, 0].$$

Using 16 collocation points in Haar wavelets discretization method the optimal value is obtained $J = 0.4220$. This value compares well with those given in [10]. The near optimal control and state variables which are obtained by the Haar wavelet discretization method are shown in Figs. 5 and 6 for $M = 16$ and $M = 32$, respectively.

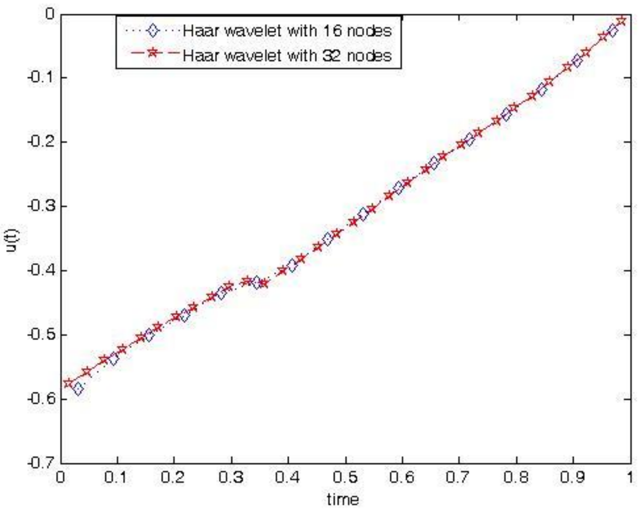

Fig. 5. The approximate optimal control input in Example 8.3.

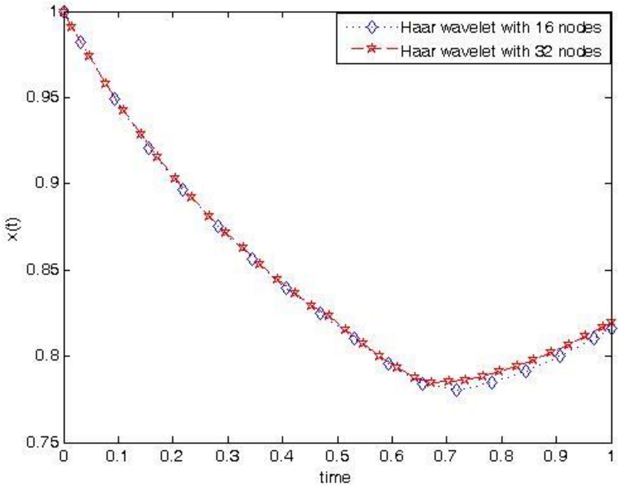

Fig. 6. The approximate optimal trajectory in Example 8.3.

9. Conclusions

In this paper using the properties of Haar wavelets, a collocation based method is presented for the resolution of optimal control governed by linear time delay systems. The given manner is based on converting the original problem to a nonlinear programming problem. One interesting advantage of the proposed method is its simplicity. The numerical results show that increasing the number of points, it is possible to improve the objective function as well as the convergence of approximate solution of the problem may lead to the exact optimal solution. Also the derived results indicate the that the proposed approach leads to find the trajectory and control functions that the corresponding objective function is better than some other methods.

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