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ENHANCING STUDENTS' ADVANCED MATHEMATICAL THINKING IN ALGEBRA THROUGH THREE WORLDS JOURNEYS ON MATHEMATICS WORKSHEETS

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Abstract

This study investigates the development of primary students' advanced mathematical thinking (AMT) skills in solving algebra problems using worksheets based on Tall's three-worlds journey of mathematics. Employing an explanatory sequential design, this research integrated quantitative and qualitative data from 40 fifth-grade students. Instruments and data analyses were validated through expert judgment and triangulation. The findings reveal that students achieved AMT skills, including mathematical representation, creativity, abstraction, and proof. Approximately 40% of students reached the natural phase, 35% progressed to the natural-to-formal phase, and 35% attained the formal-axiomatic phase. These results occurred because Tall's approach leverages children's natural tendencies to play, imagine, and explore, fostering deeper reasoning. This study impacts mathematics education by demonstrating that AMT skills can be nurtured at the primary level through structured scaffolding, enhancing students' cognitive development and problem-solving abilities in algebra.

Keywords: Advanced mathematical thinking, Algebra problems, Mathematical proof, Primary schools, Three worlds of mathematics.

1. Introduction

AMT skills align with five critical aspects of mathematics process standards: representation, connection, communication, proof and reasoning, and problem-solving [1-3]. Many reports have been well documented for mathematical thinking [4, 5]. These skills are vital for primary school students, as mandated by the Indonesian Ministry of Education and Culture through Regulation Number 5 of 2022. One essential strategy to foster these skills is the implementation of worksheets based on Tall's three-worlds journey of mathematics, designed to address whole-number arithmetic problems, a foundational element of algebra in primary mathematics education [6, 7].

In practice, mathematics learning at the primary level often remains inadequate. Students frequently struggle to translate mathematical statements into accurate concepts and procedures, and face difficulties understanding abstract concepts, either failing to fully grasp the ideas or maintaining unclear conceptual frameworks [8-11]. Additionally, students tend to replicate teacher-provided methods without independently developing new concepts [8, 12, 13]. Constructing mathematical proofs, particularly in the early stages, poses further challenges [8, 14].

This research becomes critical as mathematical proof (an AMT component) continues to be a subject of debate at the primary level [13, 15, 16]. The study explores students' AMT skills in solving algebra problems through worksheets grounded in Tall's three-worlds journey of mathematics. Employing an explanatory sequential design, this study integrates quantitative and qualitative data to comprehensively analyze students' cognitive development. The novelties of this research are: (i) adapting the three-worlds journey into worksheet design to scaffold AMT development, (ii) enhancing primary students' AMT skills systematically through this framework, and (iii) teaching algebra by bridging real-life contexts with abstract reasoning via open-ended, non-routine problems [6, 17].

2. Literature Review

Tall's three worlds of mathematics (Natural, Natural to Formal, and Formal-Axiomatic) provide a framework to develop AMT skills through progressive phases of understanding, from concrete experiences to abstract reasoning [6, 17]. In the Natural world, learning is grounded in physical manipulation and personal experience, such as using Dienes' blocks to model numbers and arithmetic operations, enhancing representation and creativity [6]. The Natural to Formal stage bridges concrete actions to symbolic representations, encouraging communication and conceptual understanding through comparative reasoning [6, 17]. Finally, the Formal-Axiomatic world emphasizes abstraction and proof, where students systematize knowledge into formal arguments accepted within the mathematical community [6, 17]. An example of the problem solving in mathematics is *"Faishal plans to visit Ragunan Zoo during the upcoming school vacation. He also intends to buy ice cream to share with his younger siblings. The Dignity Shop is the ice cream place Faishal plans to visit during his trip to Ragunan Zoo. By 10:00 AM, customers at the shop had used 35 glasses. By 6:00 PM, the usage of cups had increased fivefold. How many cups were used at 6:00 PM?"*

Figure 1 illustrates these stages, showing how algebraic problem-solving progresses from physical manipulation to symbolic and formal reasoning. The fourtier model complements this framework by diagnosing misconceptions through layers of answers, reasoning, and confidence levels. Studies affirm that integrating concrete-pictorial-abstract approaches enhances students' reasoning and proof abilities even at the primary level [15, 16].

This research applies Tall's framework in algebra problem-solving for elementary students, bridging gaps between concrete experiences and formal proof, addressing debates over AMT feasibility at this level [14, 15]. Prior studies confirm that AMT skills, including abstraction and proof, can be fostered in young learners when scaffolded appropriately [16].

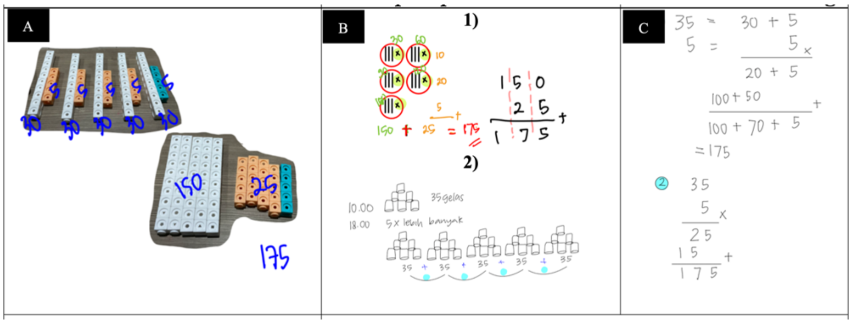

Fig. 1. AMT process through Tall's three worlds of mathematics: A) Natural, B) Natural to formal, C) Formal-axiomatic.

3. Method

This study employed an *explanatory sequential design*, integrating quantitative and qualitative approaches. Detailed information for this method is explained elsewhere [18]. We focused on examining students' AMT skills in solving algebra problems. The research involved 40 fifth-grade primary school students. Quantitative data were collected through open-ended, non-routine algebra problem worksheets grounded in Tall's three worlds of mathematics, validated by experts in mathematics education and assessment. Qualitative data were gathered from student worksheets, classroom observations, teaching modules, and reflective notes. Instrument reliability and validity were confirmed through trials with 30 students, showing high consistency (Cronbach's Alpha 0.841-0.947). Data triangulation ensured the credibility of findings.

4. Results and Discussion

Table 1 explains the rubric scores for assessing students' AMT skills based on Tall's three worlds of mathematics. The rubric categorizes student responses into Natural, Natural to Formal, and Formal-Axiomatic phases, each rated from Mediocre to Excellent.

Figure 2 explains Worksheet A, which shows student responses across the three phases. Figure 2(A) highlights the Natural phase, with 40% of students achieving the excellent category due to the effective use of concrete objects like Dienes' blocks. Figure 2(B) represents the Natural to Formal phase, where students transition from concrete representations to symbolic forms, enhancing creativity and abstraction. Figure 2(C) shows the Formal-Axiomatic phase, the most challenging, where 38% of students are in the mediocre category, reflecting difficulties in formal proof and reasoning.

Table 1. AMT's skills process rubric score.

Tall's World	AMT Skills' Components	Score Interval	Category
Natural	Finalization of fulfillment processes and accurate responses	$2 < x \leq 3$	Excellent
	Incomplete processes for completion and accurate responses	$1 < x \leq 2$	Sufficient
	Incomplete settlement processes and erroneous responses	$0 \leq x \leq 1$	Mediocre
Natural to Formal	Finalization of fulfillment processes and accurate responses	$2 < x \leq 3$	Excellent
	Incomplete processes for completion and accurate responses	$1 < x \leq 2$	Sufficient
	Incomplete settlement processes and erroneous responses	$0 \leq x \leq 1$	Mediocre
Formal-Axiomatic	Finalization of fulfillment processes and accurate responses	$2 < x \leq 3$	Excellent
	Incomplete processes for completion and accurate responses	$1 < x \leq 2$	Sufficient
	Incomplete settlement processes and erroneous responses	$0 \leq x \leq 1$	Mediocre

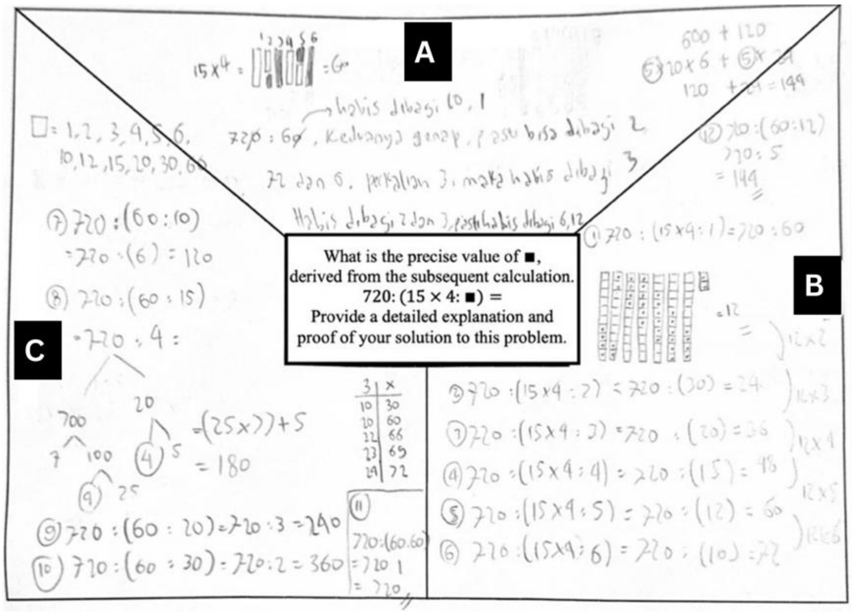

Fig. 2. Students worksheet A: (A)Natural, (B)Natural to formal, (C)Formal-axiomatic.

Figure 3 explains Worksheet B, confirming similar patterns. Figure 3(A) again emphasizes students' strong performance in the Natural phase, while Figs. 3(B) and 3(C) display the gradual shift toward formal reasoning, with varying degrees of success.

These findings confirm that primary students can engage in AMT processes through Tall’s three worlds, especially when supported by structured worksheets. The natural phase leverages students' innate play and exploration, while the formal-axiomatic phase demands higher cognitive abstraction.

These results are supported by previous research indicating that concrete-pictorial-abstract approaches foster AMT skills [13, 19-28]. Although some debate whether AMT is achievable at the primary level, this study demonstrates that structured scaffolding enables students to progress through Tall’s worlds, validating the role of guided worksheets and peer collaboration in nurturing mathematical reasoning[14, 29]. This study adds new information regarding mathematics education, as reported elsewhere [30-35].

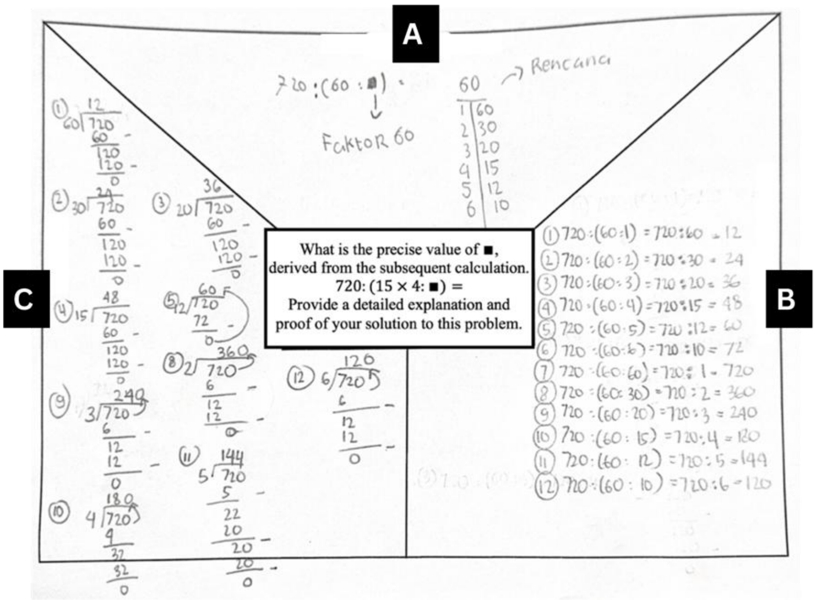

Fig. 3. Students worksheet B: (A)Natural, (B)Natural to formal, (C)Formal-axiomatic.

5. Conclusion

This study demonstrated that Tall's three worlds of mathematics, applied through structured worksheets, effectively supported primary students' AMT skills in solving algebra problems. Approximately 40% of students achieved excellence in the Natural phase, 35% in the Natural to Formal phase, and 35% in the Formal-Axiomatic phase. This success is due to the scaffolding of concrete experiences, symbolic reasoning, and formal proof, fostering creativity, abstraction, representation, and proof skills. The approach bridges concrete to abstract thinking, confirming that AMT development is achievable in elementary education.

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