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NUMERICAL SOLUTION OF BBM-BURGER EQUATION WITH QUARTIC B-SPLINE COLLOCATION METHOD

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Abstract

The present article is concerned with the numerical solution of Benjamin-Bona-Mahony-Burgers (BBM-Burger) equation by quartic B-spline collocation method. The method is based on quartic B-spline basis functions for space integration, and Crank-Nicolson formulation for time integration. Numerical examples considered by different researchers are discussed to illustrate the efficiency, robustness and reliability of the proposed method. Unconditional stability of the proposed method for BBM-Burger equation has been discussed and demonstrated by using von-Neumann method. The computed numerical solutions are in good agreement with the results available in literature as well as with exact solutions. The proposed scheme needs less storage space and execution time hence can be easily implemented to solve equations existing in various physical models.

Keywords: BBM-Burger equation, Collocation method, Thomas algorithm.

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. Most of the problems do not possess analytical solution, and so, various numerical methods have been developed to solve nonlinear equations. The numerical methods are a good means of analyzing the nonlinear equations. The process of finding an easy to apply and accurate numerical methods is a going on process.

In this paper, quartic B-spline collocation method is applied to obtain the numerical solution of Benjamin-Bona-Mahony-Burgers (BBMB) equation:

$$u_t - u_{xxt} - \alpha u_{xx} + uu_x + \beta u_x = 0 , \quad x \in [x_L, x_R] , \tag{1}$$

Nomenclatures

h	Element size
L_2	Error
L_∞	Maximum absolute error
N	Number of partitions
Q_i	B-spline basis functions
u_{xx}	Dissipative term
u_{xxt}	Dispersive term
$u(x,t)$	A real-valued function, fluid velocity in the horizontal direction
$[x_L, x_R]$	Domain partition

Greek Symbols

α	Positive constant
β	Positive constant
Δt	Time step
$\delta j(t)$	A time-dependent quantity
ϕ	Mode number

Abbreviations

BBMB	Benjamin-Bona-Mahony-Burgers
IC	Initial condition
BCs	Boundary conditions
MAEs	Maximum absolute errors
ROC	Order of convergence

where α and β are positive constants, and $u(x,t)$ is a real-valued function which represents the fluid velocity in the horizontal direction. Equation (1) describes the mathematical model of propagation of small-amplitude long waves in nonlinear dispersive media. In particular, for $\alpha=0$ Eq. (1) becomes regularized long-wave (RLW) equation:

$$u_t - u_{xxt} + uu_x + \beta u_x = 0 \tag{2}$$

Since Eq. (2) was proposed by Peregrine [1] and Benjamin–Bona et al. [2], it is referred as Benjamin–Bona–Mahony (BBM) equation. Equation (1) features a balance between the nonlinear and dispersive effects but takes no account of dissipation. The dispersive effect of Eq. (1) is the same as the BBM equation due to dispersive term u_{xxt} whereas the dissipative effect due to dissipative term u_{xx} is same as the Burgers equation

$$u_t - \alpha u_{xx} + uu_x + \beta u_x = 0 \tag{3}$$

In recent years, various numerical approaches were adopted by the researchers to solve these equations. The equation is solved by Crank-Nicolson-type finite difference method) [3], He's method [4], Exp-function method [5, 6], Adomian decomposition method [7], Homotopy analysis method [8], and linearized implicit finite difference method (LIFDM) [9]. The exact solution of the generalized BBM equation has been obtained by Chen et al. [10] using a variable-coefficient balancing-act method. The properties of the BBMB equation have been studied theoretically by many researchers, see [11-13] and the references therein.

B-Spline basis functions have great attention in approximation theory, boundary-value-problems and partial differential equation when numerical aspects are considered. Due to local compact support on a given domain, a B-Spline function of degree k is non-zero over $(k+1)$ consecutive mesh intervals at the maximum. This attribute results in a banded sparse structure of matrices that appear in interpolation and collocation problems. The collocation method with B-Spline basis functions is an efficient technique as it is easy to apply, has a programmable computation approach and it does not involve any calculation of integrals to obtain final set of equations, hence require less computational efforts in comparison to other existing methods. Various linear or nonlinear problems have been solved by using the quartic B-splines as basis functions. For instance, quartic B-spline collocation method is designed by [14] to obtain numerical solution of modified regularized long wave (MRLW) equation, and quartic B-spline Galerkin approach for KdVB equation by [15].

The paper is organized as follows. The fundamentals on quartic B-spline basis functions are discussed in Section 2. In Section 3, the numerical method is presented followed by the initial state of the unknown vectors in Section 4. The stability analysis of the method is established in Section 5. The numerical results are discussed in Section 6 to validate performance of the method. Section 7 concludes the article.

2. Description of Quartic B-spline Collocation Method

To find the solution on domain $[x_L, x_R]$ partition $x_L=x_0<x_1,\ldots,x_{N-1}<x_N= x_R$ of the space domain is considered distributed uniformly with $h=x_{i+1} - x_i =(x_R- x_L)/N$ for $i= 0, 1, 2,\ldots, N$. The set $\{Q_{-2}, Q_{-1}, Q_0, \ldots, Q_{N+1}\}$ of quartic B-splines forms a basis over $[x_L, x_R]$, where Q_i is defined by:

$$Q_i(x) = \frac{1}{h^4}\begin{cases} (x - x_{i-2})^4 & x \in [x_{i-2}, x_{i-1}), \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 & x \in [x_{i-1}, x_i), \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4 & x \in [x_i, x_{i+1}), \\ (x_{i+3} - x)^4 - 5(x_{i+2} - x)^4 & x \in [x_{i+1}, x_{i+2}), \\ (x_{i+3} - x)^4 & x \in [x_{i+2}, x_{i+3}), \\ 0 & else\ . \end{cases} \tag{4}$$

Each quartic B-spline covers five elements, each grid point is covered by five quartic B-splines.The numerical treatment for BBMB equation using the collocation method with quartic B-spline is to find an approximate solution $U(x, t)$ to the exact solution $u(x ,t)$ given by

$$U_i(x,t) = \sum_{j=i-2}^{i+1} \delta_j(t) Q_j(x), \quad i = 0,1,\ldots,N \tag{5}$$

where $\delta j(t)$ is a time-dependent quantity to be determined at each time level. The values of $Q_i(x)$ and its derivatives at the nodal points are tabulated in Table1.

Table 1. The Values of $Q_i(x)$ and its Derivatives at x_i .

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
Q_i	0	1	11	11	1	0
Q_i'	0	$-4/h$	$-12/h$	$12/h$	$4/h$	0
Q_i''	0	$12/h^2$	$-12/h^2$	$-12/h^2$	$12/h^2$	0
Q_i'''	0	$-24/h^3$	$72/h^3$	$-72/h^3$	$24/h^3$	0

From Eqs. (4) and (5) and Table 1, the values of U_i at i^{th} nodal point is derived as follows:

$$\begin{aligned} U_i &= \delta_{i-2} + 11\delta_{i-1} + 11\delta_i + \delta_{i+1} \\ hU_i' &= 4(-\delta_{i-2} - 3\delta_{i-1} + 3\delta_i + \delta_{i+1}) \\ h^2U_i'' &= 12(\delta_{i-2} - \delta_{i-1} - \delta_i + \delta_{i+1}) \\ h^3U_i''' &= 24(-\delta_{i-2} + 3\delta_{i-1} - 3\delta_i + \delta_{i+1}) \end{aligned} \tag{6}$$

3. Implementation of the Method

Equation (1) can be rewritten as

$$(u - u_{xx})_t - \alpha u_{xx} + uu_x + \beta u_x = 0 \tag{7}$$

with the boundary conditions (BCs)

$$u(x_L, t)=g_0,\ u(x_R, t)=g_1 \tag{8}$$

along with collocation boundary condition

$$u_x(x_L, t)=\mu \tag{9}$$

and initial condition (IC)

$$u(x_i, 0)=f(x_i)\ \ , x_i \in [x_L, x_R] \tag{10}$$

The condition in Eq. (9) is necessary for unique quartic B-spline solution. After discretizing the time derivative by the finite difference method, and the spatial variables and their derivatives by Crank-Nicolson scheme, Eq. (7) is rewritten as

$$\begin{aligned} &\left[\frac{(u-u_{xx})^{n+1}-(u-u_{xx})^n}{\Delta t}\right] - \alpha\left[\frac{u_{xx}{}^{n+1}+u_{xx}{}^n}{2}\right] + \\ &\qquad \left[\frac{(uu_x)^{n+1}+(uu_x)^n}{2}\right] + \beta\left[\frac{u_x{}^{n+1}+u_x{}^n}{2}\right] = 0. \end{aligned} \tag{11}$$

The nonlinear term $(uu_x)^{n+1}$ is linearized by using quasi-linearization (Bellman and Kalaba) [16] as

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u^n u_x^{n+1} - (uu_x)^n$$

Thus, Eq. (11) becomes

$$\begin{aligned} &(u^{n+1} - u_{xx}^{n+1}) + \frac{\Delta t}{2}\left[-\alpha u_{xx}^{n+1} + u^{n+1}u_x^n + u^n u_x^{n+1} + \beta u_x^{n+1}\right] = \\ &(u^n - u_{xx}^n) - \frac{\Delta t}{2}\left[-\alpha u_{xx}^n + \beta u_x^n\right] \end{aligned} \tag{12}$$

Using the approximate values of $U(x)$ and its derivatives from Eq. (6) into Eq. (12) and on simplifying, we have

$$\begin{aligned} &\delta_{i-2}^{n+1}(s - v - w) + \delta_{i-1}^{n+1}(11s - 3v + w) + \delta_i^{n+1}(11s + 3v + w) \\ &\qquad + \delta_{i+1}^{n+1}(s + v - w) = R^n(\delta_{i-2}, \delta_{i-1}, \delta_i, \delta_{i+1}) \end{aligned} \tag{13}$$

where $R^n(\delta_{i-2},\delta_{i-1},\delta_i,\delta_{i+1})$ denotes the right-hand side of Eq. (12),

$$s = 1 + \frac{\Delta t}{2} u_x^n,\ v = \frac{2\Delta t}{h}\left(u^n + \beta\right) and\ \ w = \frac{12}{h^2}\left(1 + \frac{\alpha \Delta t}{2}\right), \tag{14}$$

In this way, a system of $(N+1)$ linear equations with $(N+4)$ unknowns

$(\delta_{-2},\ \delta_{-1},\ \delta_0, \ldots\ldots,\ \delta_N,\ \delta_{N+1})$, is obtained. In order to get unique solution for the resulting system, we need three additional equations corresponding to the unknowns δ_{-2}, δ_{-1} *and* δ_{N+1}. The three additional equations are taken from BCs (8). After eliminating δ_{-2}, δ_{-1} *and* δ_{N+1}, the system is reduced into a four-diagonal matrix system of $(N+1)$ linear equations as $A\ C_N = D_N$.

where

$$A = \begin{bmatrix} y_1 & y_2 & 0 & \ldots & . & . & 0 \\ y_3 & y_4 & 4a_4 & . & . & & \\ a_1 & a_2 & a_3 & a_4 & 0 & & \vdots \\ 0 & a_1 & a_2 & a_3 & a_4 & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ & \vdots & & a_1 & a_2 & a_3 & a_4 \\ 0 & . & & 0 & z_1 & z_2 & z_3 \end{bmatrix}, C_N = \begin{bmatrix} \delta_0^{n+1} \\ \delta_1^{n+1} \\ . \\ \vdots \\ . \\ \delta_{N-1}^{n+1} \\ \delta_N^{n+1} \end{bmatrix},$$

$$D_N = \begin{bmatrix} 4R_0 + g_0(3a_1 - a_2)/2 + \mu h(22a_1 - a_2/8) \\ 4R_1 - g_0(a_1/2) - \mu h a_1/8 \\ R_2 \\ \vdots \\ \vdots \\ R_{N-1} \\ R_N - g_1 a_4 \end{bmatrix},$$

where

$$y_1 = 33a_1 - 7a_2 + 4a_3,\ \ y_2 = 7a_1 - a_2 + 4a_4,\ \ y_3 = -7a_1 + 4a_2,\ \ y_4 = -a_1 + 4a_3$$

$$z_1 = a_1 - a_4,\ \ z_2 = a_2 - 11a_4,\ \ z_3 = a_3 - 11a_4,$$

$$a_1 = s - v - w,\ a_2 = 11s - 3v + w, a_3 = 11s + 3v + w, a_4 = s + v - w,$$

This four diagonal system of matrix is solved by a modified form of well-known *Thomas algorithm* [17].The approximate solution at the required time level can be obtained repeatedly by solving the recurrence relation, once the initial vectors have been computed at the zero-level.

4. The Initial State

The initial vectors δ_i^0 are determined by substituting the approximate solution in the IC, Eq. (10) which gives a system as of $(N+1)$ linear equations:

$$\delta_{i-2}^0 + 11\delta_{i-1}^0 + 11\delta_i^0 + \delta_{i+1}^0 = f(x_i), \qquad i = 0,1,\ldots,N \tag{15}$$

with $(N+4)$ unknowns: δ^0_{-2} , δ^0_{-1} , δ^0_0 , δ^0_{N-1}, δ^0_N, δ^0_{N+1}

To eliminate the unknowns from the system of Eq. (15), we consider the following BCs at the initial level

$u_x(x_0,0)=0; \quad u_{xx}(x_0,0)=0; \quad u_x(x_N,0)=0$

From the above BCs and with Eq. (6), we get

$$\begin{aligned} &\delta_{-2}^0+3\delta_{-1}^0=3\delta_0^0+\delta_1^0 && \delta_{-2}^0=(3\delta_0^0-\delta_1^0)/2 \\ &\delta_{-2}^0-\delta_{-1}^0=\delta_0^0-\delta_1^0 && \Rightarrow\ \delta_{-1}^0=(\delta_0^0+\delta_1^0)/2 \\ &\delta_{N+1}^0=\delta_{N-2}^0+3\delta_{N-1}^0-3\delta_N^0 && \delta_{N+1}^0=\delta_{N-2}^0+3\delta_{N-1}^0-3\delta_N^0 \end{aligned} \tag{16}$$

From Eqs. (15) and (16), the resulting matrix system of (N+1) linear equations with (N+1) unknowns, written as

$$\begin{bmatrix} 18 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11.5 & 11.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 11 & 11 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 11 & 11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 11 & 11 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ \vdots \\ . \\ \vdots \\ . \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{N-2}) \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix} \tag{17}$$

The coefficient matrix of Eq. (17) is diagonally dominant and therefore is invertible, and it has a unique solution, which we obtained using the well-known *Thomas algorithm* [17].

5. Stability Analysis of the Method

The stability analysis of the proposed method is investigated by applying von-Neumann method. First, we have linearized the nonlinear term uu_x by considering u as a constant γ in equation (1). Now, rewrite the equation as

$$(u-u_{xx})_t-\alpha u_{xx}+(\gamma+\beta)u_x=0 \tag{18}$$

Discretizing the time derivative by usual *finite difference scheme* and applying *Crank-Nicolson scheme,* the equation can be written in terms of unknown time parameters δ_i*'s* using Eq. (6) as

$$\begin{aligned} &\delta_{i-2}^{n+1}(1-v-w)+\delta_{i-1}^{n+1}(11-3v+w)+\delta_i^{n+1}(11+3v+w)+\delta_{i+1}^{n+1}(1+v-w) \\ &\qquad =R^n(\delta_{i-2},\delta_{i-1},\delta_i,\delta_{i+1}) \end{aligned} \tag{19}$$

where v , w are same as defined in Section 3,

$z=\dfrac{12}{h^2}\left(1-\dfrac{\alpha\Delta t}{2}\right)$, and

$$R^n(\delta_{i-2},\delta_{i-1},\delta_i,\delta_{i+1}) = \delta^n_{i-2}(1+v-z)+\delta^n_{i-1}(11+3v+z)+ \\ \delta^n_i(11-3v+z)+\delta^n_{i+1}(1-v-z)$$

On setting,

$$b_1 = 1-v-w,\ b_2 = 11-3v+w,\ b_3 = 11+3v+w,\ b_4 = 1+v-w,$$
$$b_5 = 1+v-z,\ b_6 = 11+3v+z,\ b_7 = 11-3v+z,\ b_8 = 1-v-z,$$

Equation (19) becomes

$$\delta^{n+1}_{i-2}b_1+\delta^{n+1}_{i-1}b_2+\delta^{n+1}_{i}b_3+\delta^{n+1}_{i+1}b_4 = \delta^{n}_{i-2}b_5+\delta^{n}_{i-1}b_6+\delta^{n}_{i}b_7+\delta^{n}_{i+1}b_8 \quad (20)$$

Putting $\delta_i{}^n = \xi(t_n)\, e^{i\varphi mh}$, $i=\sqrt{-1}$ into Eq. (20) and simplifying, we have

$$\xi(t_{n+1})(e^{-2i\phi}b_1+e^{-i\phi}b_2+b_3+e^{i\phi}b_4) = \xi(t_n)(e^{-2i\phi}b_5+e^{-i\phi}b_6+b_7+e^{i\phi}b_8)$$

$$\xi(t_{n+1})(e^{-2i\phi}b_1+e^{-i\phi}b_2+b_3+e^{i\phi}b_4) = \xi(t_n)(e^{-2i\phi}b_5+e^{-i\phi}b_6+b_7+e^{i\phi}b_8) \quad (21)$$

where $\phi=\varphi h$ is the mode number, h is element size.

On simplifying Eq. (21) results in

$$\frac{\xi(t_{n+1})}{\xi(t_n)} = \frac{[(b_6+b_8)\cos\phi+b_5\cos 2\phi+b_7]+i[(-b_6+b_8)\sin\phi-b_5\sin 2\phi]}{[(b_2+b_4)\cos\phi+b_1\cos 2\phi+b_3]+i[(-b_2+b_4)\sin\phi-b_1\sin 2\phi]} = \frac{X_1+iY_1}{X_2+iY_2} \quad (22)$$

where

$$X_1 = (12+2v)\cos\phi+(1+v-z)\cos 2\phi+11-3v+z$$

$$X_2 = (12-2v)\cos\phi+(1-v-w)\cos 2\phi+11+3v+w,$$

$$Y_1 = (-10-4v-2z)\sin\phi-(1+v-z)\sin 2\phi,$$

$$Y_2 = (-10+4v-2w)\sin\phi-(1-v-w)\sin 2\phi,$$

A method is stable if $|\xi(t_{n+1})/\xi(t_n)| \le 1$. Let us assume the method is not stable, then $|\xi(t_{n+1})/\xi(t_n)| > 1$, using this condition in Eq. (22), we have

$$|X_1|^2+|Y_1|^2 > |X_2|^2+|Y_2|^2 \quad (23)$$

On simplifying Eq. (23), we have

$$8\sin^2\phi\,(w-z)[-(10+w+z)+(-2+w+z)\cos\phi] > 0 \quad (24)$$

Since $(w-z)=12\alpha\,\Delta t/h^2$ and $\sin^2\phi$ both are positive, and $(w+z)=24/h^2$. Thus, Eq. (24) implies

$$\cos\phi-1 > \frac{10+w+z}{-2+w+z}-1 = \frac{6h^2}{12-h^2} > 0,$$

which is a contradiction as $-2 \le \cos\phi - 1 \le 0$. Hence the proposed method is unconditionally stable.

6. Numerical Experiments and Discussions

To test the accuracy of the present method, four numerical examples are given in this section with the L_2 and L_∞ errors obtained by formula given by:

$$L_2 = \sqrt{\Delta x \sum_{i=0}^{N} \left| u_i^{exact} - u_i^{num} \right|^2} \ , \ L_\infty = \max_i \left| u_i^{exact} - u_i^{num} \right|$$

The numerical *order of convergence* (*ROC*) of the method is calculated by using the formula:

$$ROC = \frac{\log\left(E(N_1)/E(N_2)\right)}{\log(2)},$$

where $E(N_i)$ denotes L_∞ error norm with obtained by taking $N_i=iN$ (i=1,2) number of partitions.

Example 6.1

In this example, the BBMB equation (1) is considered for $\alpha=\beta=1$ with the initial condition

$$u(x,0)=exp(-x^2) \tag{25}$$

with BCs: (8) and with $g_0=g_1=0$, $\mu=0$

The numerical solutions are obtained over the region $[-10,10]$ at different time levels, taking Δt=0.01, N=100. The order of convergence is evaluated at t=5, 10 and is reported in Table 2. Since the exact solution is not known, to obtain the order of convergence, the maximum absolute errors (MAEs) L_∞ is obtained for different number of partitions considering the solution at N=400, as exact solution. The physical behaviour of solutions with $\alpha=\beta=1$ at different time levels $t \leq 10$ is depicted graphically in Figs. 1 and 2. Similar figures are depicted in [3].

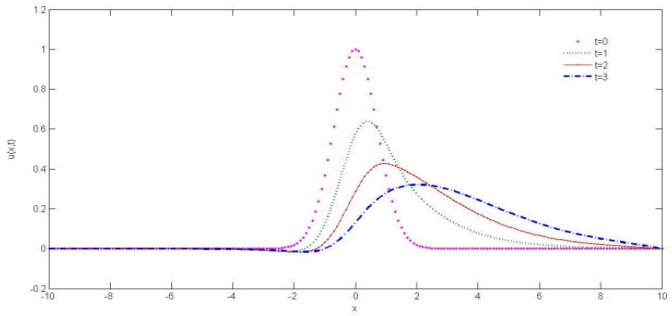

Fig. 1. The Physical Behaviour of Numerical Solutions of Example 6.1 at Different Time Levels $0 \leq t \leq 3$.

Table 2. MAEs & Order of Convergence of Example 6.1.

***t*=5**				***t*=10**		
N	L_∞	Ratio	ROC	L_∞	Ratio	ROC
50	5.73E-04	_	_	1.567E-03	_	_
100	1.37E-04	4.190	2.066	3.474E-04	4.509	2.173
200	2.80E-05	4.880	2.288	4.560E-05	7.618	2.929

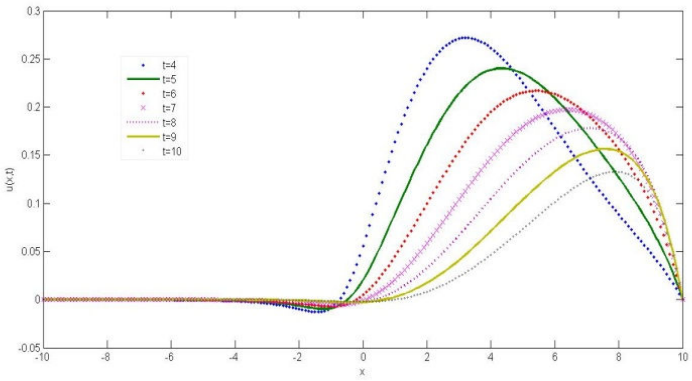

Fig. 2. The Physical Behavior of the Solution of Example 6.1at Different Time Levels $4 \leq t \leq 10$.

Example 6.2

The inhomogeneous BBMB equation over the domain [0, π] is considered as given in [3]

$$u_t - u_{xxt} - \alpha u_{xx} + uu_x + \beta u_x = \exp(-t)\left[\cos x - \sin x + \frac{1}{2}\exp(-t)\sin(2x)\right] \quad (26)$$

with initial and boundary conditions taken from the exact solution given by

$$u(x,t)=exp(-t)\ sin(x) \quad (27)$$

The results are computed for $\alpha=\beta=1$ with $\Delta t=0.01$ and different values of N. In Table 3, L_2 and L_∞ errors are computed for $N=121$ at different time-levels. In Table 4, the comparison of obtained L_2 errors with the errors due to Omrani and Ayadi [3] at $t=10$ are reported for different values of N and Δt. It is shown that with the refinement of the mesh, the results become more accurate and approaches towards the exact solutions. The physical behaviour of the solutions at different time levels $t \leq 8$ are depicted graphically in Figs. 3 and 4.

Table 3. Errors at Different Time Levels of Example 6.2.

Errors	$t=1$	$t=2$	$t=4$	$t=10$
L_∞	7.673037E-3	2.838013E-3	3.841379E-4	4.059460E-6
L_2	5.127017E-3	1.729701E-3	2.124032E-4	4.078352E-6

Table 4. L_2 Errors of Example 6.2 with $\Delta t=0.01$ at $t=10$

N	Present Method	[3]
10	1.7147E-4	0.022 E-0
20	5.6341E-5	0.005 E-0
80	7.2635E-6	3.329E-4
320	8.1631E-7	2.076E-5

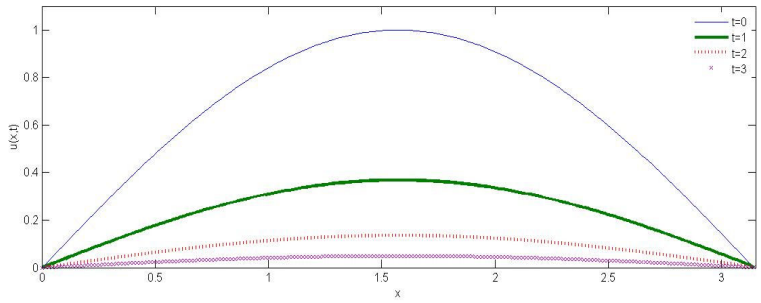

Fig. 3. The Physical Behavior of the Numerical Solutions of Example 6.2 at Different Time Levels $t \leq 3$.

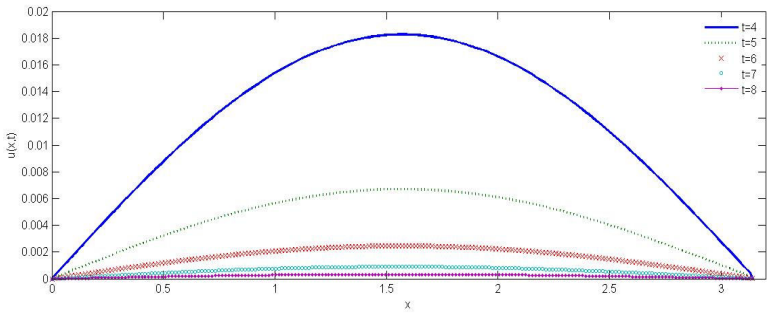

Fig. 4. The Physical Behavior of the Numerical Solutions of Example 6.2 at Different Time Levels $4 \leq t \leq 8$.

Example 6.3

The BBMB equation (1) is solved with α=0 and β=1 as considered in [6, 8], with the initial condition

$$u(x,0)=sech^2(x/4) \tag{28}$$

with appropriate boundary conditions taken from the exact solution given by Yong [10] as

$$u(x,t)=sech^2(x/4- t/3) \tag{29}$$

The numerical solutions are obtained for domain [-10, 30] at different time levels $t \leq 20$ with N=200, Δt=0.01. The L_2 and L_∞ errors at different time levels $t \leq 20$ are reported in Table 5. The comparison of the numerical results with the exact solutions is shown graphically in Fig. 5. The continuous line shows the analytical solution while the numerical solution is shown by shapes.

Table 5. Errors at Different Time Levels $t \leq 20$ in Numerical Solution of Example 6.3.

Errors	t=1	t=10	t=15	t=20
L_∞	8.268584E-4	4.103944E-4	3.389881E-4	3.164168E-4
L_2	5.510903E-4	5.504361E-4	5.894528E-4	6.000788E-4

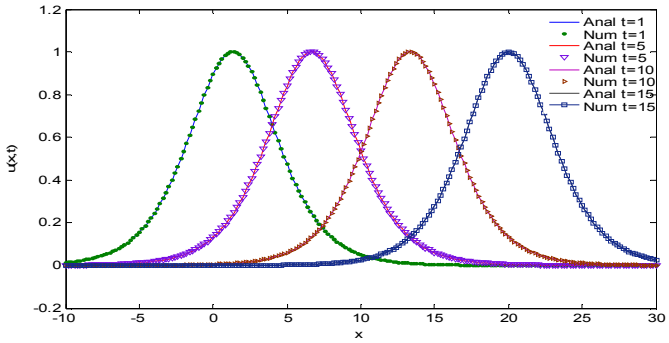

Fig. 5. The Comparison of the Numerical Solutions of Example 6.3 with Exact Solution at Different Time Levels $t \leq 15$.

Example 6.4

The BBMB equation (1) with α=0 and β=1 is solved with the initial and boundary conditions taken from the exact solution given by

$$u(x,0)=3c\ sech^2(k\ (x-\ a-\ vt)) \tag{30}$$

The solution represents a single solitary wave of amplitude $3c$ and width $k=(1/2)\sqrt{(c/v)}$, initially centered at a, where $v=1+c$ is the wave velocity. The numerical solutions are obtained for domain [-40, 60], at different time levels $t \leq 20$ taking the parameter values c=0.1, Δx=0.125, Δt=0.1 and a=0.The L_2 and L_∞ errors are compared with those obtained by Kutlay and Esen [9] at different time-levels in Table 6. The comparison of the numerical results with the exact solutions is shown graphically in Fig. 6. It is evident that the present method produces good results.

Table 6. Comparison of Errors in Example 6.4 with errors in [9] at $t \leq 20$

Schemes	Errors	t=4	t=8	t=12	t=16	t=20
Present	L_∞	0.15E-4	0.33E-4	0.49E-4	0.64E-4	0.78E-4
LIFDM[9]		0.05E-3	0.09E-3	0.14E-3	0.18E-3	0.21E-3
Present	L_2	0.42E-4	0.33E-4	0.13E-3	0.16E-3	0.20E-3
LIFDM[9]		0.12E-3	0.23E-3	0.34E-3	0.45E-3	0.55E-3

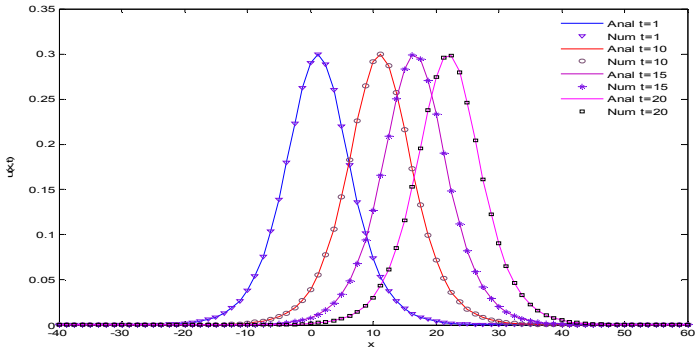

Fig. 6. The Comparison between Numeric and Exact Solution of Example 6.4 at Different Time Levels $t \leq 20$.

7. Conclusions

In this article, the numerical solution of the nonlinear BBM-Burger equation is obtained by using collocation method with quartic B-splines as basis functions. The performance of the method has been evaluated by considering four test problems and calculating the error norms for different time levels and comparing results with those available in literature. The rate of convergence is also calculated and found to be second order convergent. The method successfully provides very accurate solutions taking different parameters. The results show that error decreases with the increase in time. Further, with the refinement of the mesh, the numerical results become more accurate and approach towards the exact solutions. The results are found in good agreement with the analytical solutions and better with the solutions available in the literature. Using linear stability analysis, it is shown that the proposed method is unconditionally stable therefore there is no restriction on the grid size, but we should choose step lengths in such a way that the accuracy of the method is not degraded. The proposed method is easy to implement hence can be applied to solve various linear and nonlinear physical models.

References

1. Peregrine, D.H. (1996). Calculations of the development of an undular bore. *Journal of Fluid Mechanics,* 25(2), 321-330.
2. Benjamin, T.B.; Bona, J.L.; and Mahony, J.J. (1972). Model equations for waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society A*, 272 (1220) 47-78.
3. Omrani, K.; and Ayadi, M. (2008). Finite difference discretization of the Benjamin-Bona-Mahony-Burgers equation. *Wiley Inter Science*, 24(1), 239-248.
4. Tari, H.; and Ganji, D.D. (2007). Approximate explicit solutions of nonlinear BBMB equations by He's methods and comparison with the exact solution. *Physics Letters A*, 367(1-2) 95–101.
5. Ganji, Z.Z.; Ganji, D.D.; and Bararnia, H. (2009). Approximate general and explicit solutions of nonlinear BBMB equations by Exp-Function method. *Applied Mathematical Modelling*, 33(4) 1836-1841.
6. El-Wakil, S.A.; Abdou, M.A.; and Hendi, A. (2008). New periodic wave solutions via Exp-function method. *Physics Letters A,* 372(6), 830-840.
7. Al-Khaled, K. (2005). Approximate wave solutions for generalized Benjamin–Bona–Mahony–Burgers equations. *Applied Mathematics Computions*, 171(1) 281-292.
8. Fakhari, A.; Domairry, G.; and Ebrahimpour (2007). Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with the exact solution. *Physics Letters A,* 368(1-2) 64-68.
9. Kutlay, S.; and Esen, A. (2006). A finite difference solution of the regularized long-wave equation. *Hindawi Publishing Corporation Mathematical Problems in Engineering*, Article ID 85743, DOI 10.1155/MPE/2006/85743.
10. Chen, Y.; Li, B.; and Zhang, H. (2005). Exact solutions of two nonlinear wave equations with simulation terms of any order. *Communications in Nonlinear Science and Numerical Simulation*, 10, 133-138.

11. Yin, H.; Zhao, H.; and Kim, J. (2008). Convergence rates of solutions toward boundary layer solutions for generalized Benjamin-Bona-Mahony-Burgers equations in the half-space. *Journal of Differential Equations*, 245(11), 3144-3216.
12. Zhao, H.J.; and Xuan, B.J. (1997). Existence and convergence of solutions for the generalized BBM–Burgers equations with dissipative term. *Nonlinear Anal. TMA*, 28(11), 1835-1849.
13. Zhang, L.H. (1995). Decay of solutions of generalized Benjamin–Bona–Mahony–Burgers equations in n-space dimensions. *Nonlinear Analysis* 25(12), 1343-1369.
14. Fazal-i-Haq;Siraj-ul-Islam; and Tirmizi, I.A. (2010). A numerical technique for solution of the MRLW equation using quartic B-splines. *Applied Mathematical Modelling*, 34(12), 4151-4160.
15. Saka, B.; and Dag, I. (2009). Quartic B-spline Galerkin approach to the numerical solution of the KdVB equation. *Applied Mathematics and Computation,* 215(12), 746-758.
16. Bellman R. E.; and Kalaba, R. E. (1965). *Quasi-linearization and nonlinear Boundary-Value problems*. American Elsevier Publishing Company, Inc., New York.
17. Von Rosenberg, D.U. (1969). *Methods for solution of partial differential equation.* 113, American Elsevier Publishing Inc., New York.