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SIX STAGES OF THE MATHEMATICAL THINKING MODEL IN SOLVING PROBLEMS ON A SQUARE NUMBER PATTERN

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Abstract

This study aims to analyse the application of problem sets designed using the six stages of the mathematical thinking model in solving square number pattern problems and to examine students' performance across each stage. A quantitative descriptive approach was employed involving 24 eighth-grade junior high school students. Student achievement was analysed based on indicators corresponding to the six stages of mathematical thinking. The findings indicate that, on average, students were able to solve the problems effectively, with most indicators reaching the good category. This outcome suggests that students were capable of engaging with each stage of mathematical thinking, although variations occurred across indicators. The results imply the need for an initial diagnostic assessment to identify students' strengths and weaknesses at each stage of mathematical thinking before instruction. Such diagnostics can support more targeted explanations and contribute to more efficient use of instructional time.

Keywords: Mathematical thinking model, Problem solving, Six stages, Square number pattern, Student.

1. Introduction

Mathematical thinking is a fundamental component of mathematical proficiency and plays a critical role in developing students' reasoning, problem-solving abilities, and conceptual understanding in the 21st century [1, 2]. Through mathematical thinking, students are able to interpret situations, construct representations, and solve problems using logical reasoning, creativity, and reflection [3]. In mathematics education, mathematical thinking functions not only as a learning outcome but also as a process that supports deeper understanding and sustained intellectual development [4]. Nevertheless, many students continue to experience difficulties in developing coherent and reflective mathematical thinking, particularly when confronted with non-routine problems that demand abstraction and conceptual integration.

Several theoretical frameworks have been proposed to describe the stages of mathematical thinking. Mathematical thinking has been characterized as a process of conjecturing, justifying, and critiquing ideas through dialectical reasoning [5]. Other perspectives emphasize transitions from concrete representations to abstraction through reflection and reification [6] or describe mathematical thinking as a recursive process involving specialization, generalization, conjecture, and justification [7]. Although these frameworks contribute substantially to understanding how mathematical ideas develop, they are often applied in fragmented or linear ways. This limitation becomes evident in topics such as square number patterns, which require students to integrate visual, numerical, and symbolic reasoning.

Therefore, this study aims to describe the six stages of the mathematical thinking process in solving problems related to square number patterns using a quantitative descriptive approach. Student learning outcomes are analysed through questions designed according to the characteristics of each stage of mathematical thinking. The novelty of this study lies in: (i) systematically describing the six stages of mathematical thinking, (ii) analysing students' responses to problems structured around these stages, and (iii) presenting examples of square number pattern problems aligned with each stage of mathematical thinking.

2. Literature Review

A Square Number is a number that results from multiplying an integer by itself. In a visual pattern, these numbers always form a perfect square shape. Figure 1 explains that the area of a square or the number of elements in a pattern increases quadratically, not by one, and is therefore often used in mathematics lessons about number patterns, quadratic functions, or the concept of square area. The meaning is as follows: Each numbered image (1, 2, 3, 4, 5) represents a value of n. Each value of n forms a square of size $n \times n$. $n = 1 \rightarrow 1$ square ($1 \times 1 = 1$), $n = 2 \rightarrow 4$ squares ($2 \times 2 = 4$), $n = 3 \rightarrow 9$ squares ($3 \times 3 = 9$), $n = 4 \rightarrow 16$ squares ($4 \times 4 = 16$), $n = 5 \rightarrow 25$ squares ($5 \times 5 = 25$). The arrow on the right-side indicating $n \rightarrow n^2$ confirms that the number of squares is always the square of n.

Table 1 shows the square number pattern based on the six stages of mathematical thinking. Through these six stages, students not only memorize the square number formula but also understand the meaning of the square number pattern, connect mathematics with visuals (squares), and develop systematic and reflective skills. Figure 2 shows six stages of mathematical thinking that synthesize and expand upon the previous frameworks. These six stages (Conceptualization, Connection, Conjecturing, Generalization, Justification, and Reflection) represent a dynamic and recursive process students go through when constructing, linking, validating, and internalizing mathematical ideas in problem solving [8].

Conceptualization refers to the formation of a mental representation of the problem context, while connection involves linking new information to existing knowledge structures [9]. Conjecture refers to the creative process of generating tentative conjectures or hypotheses, followed by generalization, which is the process of abstracting relationships into more general principles. Justification involves logical reasoning and proof of the ideas developed, while reflection represents a metacognitive evaluation of the overall thinking process. Together, these six stages encompass both the cognitive and metacognitive dimensions of mathematical thinking and provide a comprehensive analytical lens for examining students' reasoning processes.

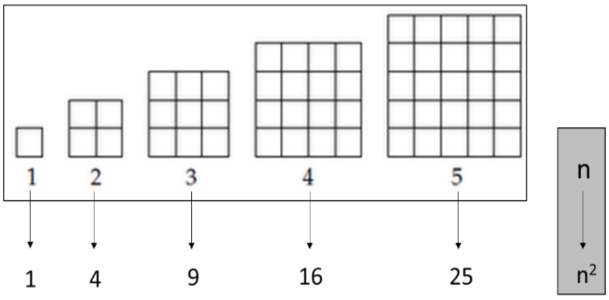


Fig. 1 Square number pattern.

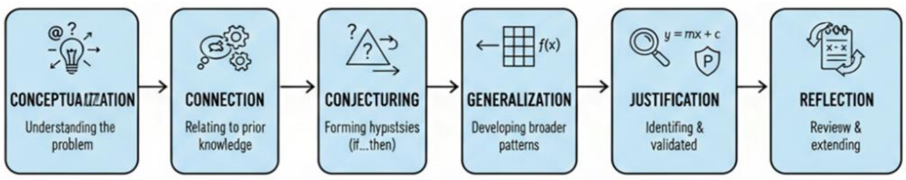


Fig. 2. Six stages of mathematical thinking.

3. Methodology

This study employed a quantitative descriptive method to examine students' mathematical thinking on square number patterns. Descriptive mathematics problems were developed based on a six-stage mathematical thinking model and administered to 24 eighth-grade students at SMPIT Al-Muhsinin, Cirebon, West Java, during the 2025/2026 academic year. Data were collected through individual written problem-solving tests and think-aloud oral assessments conducted simultaneously. Students freely expressed their thinking without guidance, academic assistance, or strategy restrictions within approximately 30 minutes. Additional interviews were conducted after testing to clarify students' responses, ensuring a neutral, non-judgmental research atmosphere.

4. Results and Discussion

Figure 3 is a question developed and used in this study; the question consists of one question with six sub-questions that are adjusted to six aspects of the stages of mathematical thinking. Indicators in the initial understanding of patterns to interpret the structure of number patterns, then in the concept connection linking patterns with the concept of squares, for the mathematical conjecture stage proposing a formula conjecture, the general rule stage is asked to write a general formula, followed by mathematical reasoning by giving reasons for the truth of the formula and metacognitive awareness to evaluate the thinking process.

Table 1. Square number patterns based on the six stages of mathematical thinking.

Mathematical Thinking Stage	Definition	Examples of Square Number Patterns
Conceptualization	The process of understanding the basic concepts of a mathematical problem or idea.	Students observe that a 1×1 square has 1 square, a 2×2 square has 4 squares, and a 3×3 square has 9 squares. Students understand that this pattern relates to squares and area.
Connection	Connecting new concepts to previously learned mathematical knowledge or concepts.	Students relate the pattern of the number of squares to the multiplication operation (n × n) and the previously learned concept of the area of a square.
Conjecturing	Make a guess or hypothesis based on the pattern or observation made	Students guess that the number of squares in the nth square is always equal to n times n. For example, for n = 6, the number of squares is 36.
Generalization	Develop a general rule or formula that applies to all cases in the pattern.	Students conclude that the general formula for the square number pattern is $Un = n^2$
Justification	Provide a reason or proof of why the formula or conclusion is correct.	Students explain that the nth square has n rows and n columns, so the total number of squares is $n \times n = n^2$.
Reflection	Review the thought process, the results obtained, and their possible applications in other contexts.	Students reflect on how square number patterns can be used to calculate the area of a square garden or the number of tiles on a square floor.

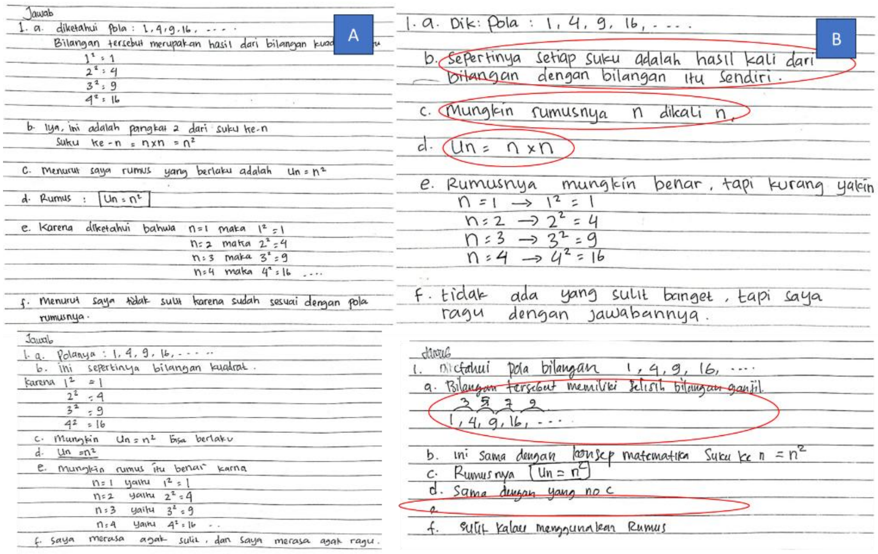
PROBLEM-SOLVING TEST INSTRUMENT
SQUARED NUMBER PATTERNS – GRADE VIII SMPIT AL-MUHSININ

Instructions:
Work through the following questions carefully. Write down how you think about solving the questions.

QUESTION:
Consider the following number pattern:
1, 4, 9, 16, 25, …
a. What do you notice about this number pattern?
b. Does this pattern remind you of any mathematical concepts? Explain.
c. What formula do you think might apply to the nth term?
d. Write the general formula for the nth term of this pattern.
e. Explain why you think this formula is correct.
f. Which part of this problem did you find most difficult or most helpful?

Fig. 3. Square number pattern questions.

Figure 4 shows some examples of student answers, where Fig. 4(A) shows answers that are in accordance with the concept, while Fig. 4(B) shows answers that are different from the explanation and sequence of concepts, and there tend to be some that are not correct. This is also in accordance with spontaneous answers based on think-aloud from students, which were based on the indicators consisting of: explaining the meaning of the pattern, connecting the index and square, stating conjectures, stating general rules, checking examples, and stating doubts/confidence. The answers given by students when interviewed and testing spontaneous answers were similar or the same as the test answers they wrote on paper; the average level of error from students occurred when they had to explain the reasons for the correctness of the formula.


A

Jawab
1. a. diketahui pola : 1, 4, 9, 16, ...
Bilangan tersebut merupakan hasil dari bilangan kuad
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
b. Iya, ini adalah pangkat 2 dari suku ke-n
Suku ke-n = n×n = n^2
c. Menurut saya rumus yang berlaku adalah $Un = n^2$
d. Rumus : $\boxed{Un = n^2}$
e. Karena diketahui bahwa n=1 maka $1^2 = 1$
n=2 maka $2^2 = 4$
n=3 maka $3^2 = 9$
n=4 maka $4^2 = 16$ . . . .
f. Menurut saya tidak sulit karena sudah sesuai dengan pola rumusnya.

Jawab
1. a. Polanya : 1, 4, 9, 16, ...
b. ini sepertinya bilangan kuadrat.
karena $1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
c. Mungkin $Un = n^2$ bisa berlaku
d. $Un = n^2$
e. mungkin rumus itu benar karna
n=1 yaitu $1^2 = 1$
n=2 yaitu $2^2 = 4$
n=3 yaitu $3^2 = 9$
n=4 yaitu $4^2 = 16$ . .
f. saya merasa agak sulit, dan saya merasa agak ragu.

B

1. a. Dik: Pola : 1, 4, 9, 16, ...
b. Sepertinya setiap suku adalah hasil kali dari bilangan dengan bilangan itu sendiri.
c. Mungkin rumusnya n dikali n.
d. Un = n × n
e. Rumusnya mungkin benar, tapi kurang yakin
n = 1 → $1^2 = 1$
n = 2 → $2^2 = 4$
n = 3 → $3^2 = 9$
n = 4 → $4^2 = 16$
f. tidak ada yang sulit banget, tapi saya ragu dengan jawabannya.

Jawab
1. Diketahui pola bilangan 1, 4, 9, 16, ...
a. Bilangan tersebut memiliki selisih bilangan ganjil.
3 5 7 9
1, 4, 9, 16, . . .
b. ini sama dengan konsep matematika Suku ke n = n^2
c. Rumusnya $\boxed{Un = n^2}$
d. Sama dengan yang no c
e.
f. sulit kalau menggunakan Rumus

Fig. 4. Answer students: a) same answer for students 1 and 2, b) different answer student 3 and 4.

In the conceptualization stage, students attempt to build an initial understanding of the given number pattern. This stage focuses on how students interpret the structure of the pattern before forming a formal mathematical formula or model.

The students analysed at this stage were S1, S2, S3, and S4. Student S1 demonstrated a strong conceptual understanding by linking each number in the pattern to the result of multiplying the number by itself, as seen in his statement: "$1 = 1 \times 1$, $4 = 2 \times 2$, and $9 = 3 \times 3$." This statement indicates that S1 has developed a relational understanding of the square pattern, not simply observing the increase in value. Student S4 demonstrated a general initial understanding. He identified that the numbers in the pattern "get bigger and in sequence," but had not yet explicitly linked this to the concept of squares. This is reflected in his statement: "The numbers get bigger and in sequence." Student S2 immediately recognized the pattern as a square number but provided a very brief explanation: "This pattern is a square number." Meanwhile, S3 demonstrated a still-uncertain conceptualization, stating: "It looks like a square, but I'm not sure yet." Findings at this stage indicate that students' conceptualizations range across a broad spectrum, from a strong relational understanding to an unstable intuitive understanding. This stage provides an important foundation for the continuation of mathematical thinking processes.

The connection stage illustrates how students relate their initial understanding of patterns to previously learned mathematical concepts. The students analysed at this stage were S1, S2, and S3. Student S2 demonstrated clear connection skills by connecting the number of a term and its quadratic form: The 1st term = 1^2, the 2nd term = 2^2. This statement indicates that S2 has been able to connect the index of the term with the symbolic representation of the square. Student S1, who previously only observed the growth of numbers, begins to build connections through concrete examples: 1 is 1^2, 4 is 2^2. This indicates a shift from numerical observation to conceptual understanding [10, 11].

The generalization stage is characterized by students' ability to formulate a general rule that applies to all terms in the pattern. Students S1, S2, and S3 demonstrated this ability by writing the general formula for the nth term. Student S1 explicitly stated: "The general formula for the nth term is $Un = n^2$." Student S2 added a condition to his generalization: "If n is a natural number, $Un = n^2$." Students S1 and S2 wrote the generalization briefly without further explanation: "$Un = n^2$." These results indicate that generalizations can appear in formal form, but the level of conceptual meaning that accompanies them varies among students. Student S4 builds connections verbally without using formal symbols, as seen in her statement: "The nth number multiplied by itself." Although not written in algebraic notation, this statement indicates that the connection between the index and the value of the term is beginning to form [12]. Overall, the connection stage serves as a bridge between intuitive understanding and more formal mathematical representation [13].

This is proven by the recapitulation of the results of the mathematical thinking process test where the average of 24 students was 7.1 with a moderate category, with details of the average value per indicator being conceptualization of 1.75 (strong emergence), connection 1.54 (strong emergence), conjecturing 0.63 (limited emergence), generalization 1.54 (strong emergence), justification 1.13 (strong emergence), reflection 0.52 (limited emergence). Overall, the research findings indicate that students' mathematical thinking processes develop through interrelated and not always linear stages. Conceptualization and connection play a crucial role in building the foundation of thinking, while justification is the most challenging stage [14, 15]. Reflection helps students recognize the strengths and limitations of their understanding. These findings provide empirical support for the six-stage model of mathematical thinking used in this study. Finally, this adds new information regarding mathematics education, as reported elsewhere [16-20].

5. Conclusion

This study examined the use of problem sets designed according to the six stages of the mathematical thinking model in solving square number pattern problems. Using a quantitative descriptive approach involving eighth-grade students, the findings indicate that students were generally able to engage with each stage of mathematical thinking and solve the problems effectively. However, variations across indicators suggest that not all stages developed evenly. These results highlight the importance of conducting an initial diagnostic assessment to identify students' strengths and weaknesses at each stage of mathematical thinking. Such diagnostics can support more focused instruction, improve conceptual understanding, and ensure more effective use of learning time.

References

1. Abidin, Z.; Herman, T.; Wahyudin, W.; and Farokhah, L. (2025). Bibliometric analysis using Vosviewer with Publish or Perish of computational thinking and mathematical thinking in elementary school. *ASEAN Journal for Science Education*, 4(1), 7-16.
2. Agustin, N.D.; Sumantri, M.S.; and Marini, A. (2026). Fostering advanced mathematical thinking through Islamic ethical integration in elementary classrooms to support sustainable development goals (SDGs). *ASEAN Journal of Educational Research and Technology*, 5(2), 181-200.
3. Schoenfeld, A.H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Journal of Education*, 196(2), 1-38.
4. Sitorus, J. (2016). Students' creative thinking process stages: Implementation of realistic mathematics education. *Thinking Skills and Creativity*, 22, 111-120.
5. Confrey, J.; and Costa, S. (1996). A critique of the selection of "Mathematical objects" as a central metaphor for advanced mathematical thinking. *International Journal of Computers for Mathematical Learning*, 1(2), 139-168.
6. Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: Transitions from arithmetic to algebra. *Educational Studies in Mathematics*, 36(3), 219-245.
7. Lesseig, K. (2016). Conjecturing, generalizing, and justifying: Building theory around teacher knowledge of proving. *International Journal for Mathematics Teaching and Learning*, 17(3), 1-31.
8. Stylianou, D.A. (2013). An examination of connections in mathematical processes in students' problem solving: Connections between representing and justifying. *Journal of Education and Learning*, 2(2), 23-35.
9. Charles, R.I.; and Carmel, C.A. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education*, 7(3), 9-24.
10. Trundle, K.C.; and Bell, R.L. (2010). The use of a computer simulation to promote conceptual change: A quasi-experimental study. *Computers and Education*, 54(4), 1078-1088.

11. Merenluoto, K.; and Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction*, 14(5), 519-534.
12. Asaishi, T. (2011). An analysis of the terminological structure of index terms in textbooks. *Procedia-Social and Behavioral Sciences*, 27, 209-217.
13. Pape, S.J.; and Tchoshanov, M.A. (2001). The role of representation (s) in developing mathematical understanding. *Theory into Practice*, 40(2), 118-127.
14. Kuhn, D. (1999). A developmental model of critical thinking. *Educational Researcher*, 28(2), 16-46.
15. Kallio, E. (2011). Integrative thinking is the key: An evaluation of current research into the development of adult thinking. *Theory and Psychology*, 21(6), 785-801.
16. Bendol, R.L.; and Dalayap Jr.; R.H. (2025). Structural equation modelling of factors influencing confidence in mathematics. *ASEAN Journal of Educational Research and Technology*, 4(2), 107-124.
17. Jose, M.T.N.S. (2022). Factors that affect the performance of selected high school students from the third district of Albay in International Mathematics Competitions. *ASEAN Journal for Science Education*, 1(1), 9-16.
18. Dermawan, R.; Muktiarni, M.; and Mupita, J. (2022). Efforts to increase the interest of junior high school students in mathematics lessons using the TikTok learning tool. *ASEAN Journal for Science Education*, 1(2), 81-88.
19. Lagcao, Y.G.D.; Dechavez, J.P.A.D.; Goleng, D.J.G.; Lagca, Y.G.D.; Tangkli, K.Y.M.; and Vicera, W.J.C. (2023). Math readiness and its effect on the online academic performance of science, technology, engineering, and mathematics students. *ASEAN Journal for Science Education*, 2(1), 33-38.
20. Awofala, A.O.A.; and Olaniyi, A.O. (2023). Assessing teachers' formative evaluation strategy as related to senior secondary school students' achievement in mathematics. *ASEAN Journal for Science Education*, 2(2), 77-86.