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MANIFOLD BASED CONTROLLER (MBC) DESIGN FOR LINEAR SYSTEMS

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Abstract

A linear feedback controller is usually designed based on many approaches like poles placement, linear quadratic regulator and others. In this work the linear feedback controller is designed based on creating an output function named manifold function and then design the controller to regulate this function to zero level and keep it there for all future time. On the other hand the manifold function, is derived provided that the system dynamics is minimum phase with respect to it. This will ensure the asymptotic stability for the whole system. Furthermore, the manifold function zero level will divide the state space into two halves and keeps the state in one of them depending on its initial condition. This feature is helpful for the case of constrained states system. A linearized model for a container crane is utilized as a case study for the application of the manifold based controller. The simulation results showed the effectiveness of the manifold based approach in designing a linear controller to constrain the sway angle within a certain limit during load transportation. In addition the designed controller was robust for the variation of the load mass.

Keywords: Manifold based controller, Manifold Function, Crane Container.

1. Introduction

For a controllable linear time invariant system,

$$\dot{x} = Ax + B \tag{1}$$

where, $x \in \mathcal{R}^n$, $A \in \mathcal{R}^{n \times n}$,$B \in \mathcal{R}^{n \times m}$ and $u \in \mathcal{R}^m$, a linear feedback control of the form;

$$u = -Kx \tag{2}$$

where $K \in \mathcal{R}^{\mathrm{m \times n}}$ is the constant gain matrix is usually utilized to asymptotically stabilize the system dynamics. The matrix K can be determined by

Nomenclatures

A	System plant matrix
A_n	Nominal system plant matrix
B	Control distribution matrix
D	Manifold linear mapping matrix
f_x	Horizontal force acting on the trolley
K	Feedback constant gain matrix
k_u	DC motor constant
L	Lagrangian function
l	Rope length, m
m_L	Load mass, kg
m_T	Trolley mass, kg
q	Generalized coordinate
	Manifold function
T	Kinetic energy function, J
	Nonsingular transformation
T_{load}	Load mass kinetic energy function
T_{troll}	Trolley kinetic energy function
u	Control action
V	Potential energy function, J
x	State variable vector
	Trolley position
x_L, y_L	x- and y-axes load Cartesian coordinates
z	Regular form state variable vector

Greek Symbols

ΔA	Uncertainty in system plant matrix
θ	Sway angle, deg.

different design methods. Among these methods the poles placement design and the linear quadratic regulation (LQR) are the most widely used. For all design approach the closed loop system

$$\dot{x} = (A - BK)x \tag{3}$$

where, the closed loop poles are negative roots. This feature will ensure the asymptotic stability, but the system response characteristics depend on the elements of matrix K. Therefore, when the negative roots property is preserved for the closed loop system, then by changing the elements of the matrix K, according to a certain methodology, may improve the system response characteristics from some points of view. This is, in fact, the idea behind the present work.

In some control theories, like in sliding mode control theory [1], the controller task is to direct the state toward a certain surface in state space that passes through the origin. When the state is at this surface, it goes asymptotically to the origin where this behavior represents an essential property for this surface. In a general setting this surface is named as a **manifold** which may represent a curve, a surface or a hyper surface with a certain properties [2]. The manifold notion also appears in the nonlinear control theories that use the differential geometric approach in the controller and the observer design [3, 4].

In the present work the manifold concept, as stated above, will be used in deriving the control law; namely by deriving the manifold equation and regard it as an output. The controller task is to regulate this output to a zero level. The zero level is the required manifold such that the system state will asymptotically go to the origin when initiated and on it. Hence a new K matrix will be derived for stabilizing system, Eq. (1).

As a case study, the container crane system model will be used, where the proposed controller based manifold will be applied to translate the load to the required position with a small sway angle.

2. Manifold Based Control Design

The first step presented in this section is to determine a nonsingular matrix transform that will transform the system dynamics, Eq. (1), to a form known as a regular form (RF) [5]. The regular form will be used subsequently in this section in designing the manifold function. Let the following nonsingular transformation be utilized for transforming the system to a RF:

$$z = Tx \tag{4}$$

where $T \in \mathcal{R}^{\mathrm{n}\times\mathrm{n}}$. The dynamical system, Eq. (1) now becomes:

$$\dot{z} = TAT^{-1}z + TBu \tag{5}$$

and the matrix TB is wanted to take the following form:

$$TB = \begin{bmatrix} O \\ \tilde{B} \end{bmatrix} \tag{6}$$

where $\tilde{\mathrm{B}} \in \mathcal{R}^{\mathrm{m}\times\mathrm{m}}$ and $0 \in \mathcal{R}^{(\mathrm{n-m})\times\mathrm{m}}$ (the zero matrix). In reference [6], the above condition, Eq. (6), was used to transform the dynamical system to a certain form known as a regular form. The regular form can also be used to design a controller based on Backstepping approach [7]. Moreover, in the reduced order observer design, a condition similar to condition, Eq. (6), was used to decouple the disturbance from a subsystem represents the dynamics of the unmeasured states [8].

Now the transformation matrix T that will satisfy condition, Eq. (6), may be taken as follows:

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{7}$$

where $T_1 \in \mathcal{R}^{(\mathrm{n-m})\times\mathrm{n}}$ and $T_2 \in \mathcal{R}^{\mathrm{m}\times\mathrm{n}}$ and both of them has full rank. Hence, to satisfy Eq. (6), we have

$$T_1 B = O \tag{8}$$

and T_1 may be decomposed as follows:

$$T_1 = [T_{11} \quad T_{12}] \tag{9}$$

where $T_{11} \in \mathcal{R}^{(\mathrm{n-m})\times(\mathrm{n-m})}$ and $T_{12} \in \mathcal{R}^{(\mathrm{n-m})\times\mathrm{m}}$. Let T_{11} equal to the identity matrix $I \in \mathcal{R}^{(\mathrm{n-m})\times(\mathrm{n-m})}$, also let the matrix B to decomposed as

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{10}$$

where $B_1 \in \mathcal{R}^{(n-m)\times m}$ and $B_2 \in \mathcal{R}^{m\times m}$, then from Eqs. (8), (9), and (10) we get

$$T_{11}B_1 + T_{12}B_2 = B_1 + T_{12}B_2 = O \tag{11}$$

Solving for T_{12} in Eq. (11), we obtain

$$T_{12} = -B_1B_2^{-1} \tag{12}$$

where B_2 has a full rank (rank $B_2 = \mathrm{m}$). Therefore the matrix T_1 becomes

$$T_1 = \begin{bmatrix} I_{(n-m)\times(n-m)} & -B_1B_2^{-1} \end{bmatrix} \tag{13}$$

Now to get the transformation matrix T the image of the matrix T_2 must lies in the null space of matrix T_1, i.e.,

$$Img(T_2) \subset N(T_1) \tag{14}$$

This also will ensure that the transformation matrix T to have a full rank as required. To compute T_2, we first decompose it as follows:

$$T_2 = [T_{21} \quad I_{m\times m}],\ \ T_{21} \in \mathcal{R}^{m\times(n-m)} \tag{15}$$

Then the following product, which it is equivalent to condition, Eq. (14), must be satisfied

$$T_2T_1^T = 0 \quad \Rightarrow [T_{21} \quad I_{m\times m}]\begin{bmatrix} I_{(n-m)\times(n-m)} & -B_1B_2^{-1} \end{bmatrix}^T = 0 \tag{16}$$

or

$$T_{21} = (B_1B_2^{-1})^T \tag{17}$$

Eventually, the transformation matrix T is equal to

$$T = \begin{bmatrix} I_{(n-m)\times(n-m)} & -B_1B_2^{-1} \\ (B_1B_2^{-1})^T & I_{m\times m} \end{bmatrix} \tag{18}$$

The matrix T is non singular if the following condition holds:

$$det(T) = det[I_{m\times m} + (B_1B_2^{-1})^T B_1B_2^{-1}] \neq 0 \tag{19}$$

The dynamical system, Eq. (1), is transformed now to the following form:

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = TAT^{-1}z + TBu = \begin{bmatrix} \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 \\ \tilde{A}_{21}z_1 + \tilde{A}_{22}z_2 \end{bmatrix} + \begin{bmatrix} O \\ \tilde{B} \end{bmatrix} u \tag{20}$$

or

$$\dot{z}_1 = \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 \tag{21-a}$$

$$\dot{z}_2 = \tilde{A}_{21}z_1 + \tilde{A}_{22}z_2 + \tilde{B}u \tag{21-b}$$

where $z_1 \in \mathcal{R}^{n-m}$, $z_2 \in \mathcal{R}^{m}$, $TAT^{-1} = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$,

$\tilde{A}_{11} \in \mathcal{R}^{(n-m)\times(n-m)}$, $\tilde{A}_{12} \in \mathcal{R}^{(n-m)\times m}$, $\tilde{A}_{21} \in \mathcal{R}^{m\times(n-m)}$ and $\tilde{A}_2 \in \mathcal{R}^{m\times m}$.

Now let us defined the following linear map (it may be named here as manifold function $q(z)$):

$$q(z) = Dz = DTx \tag{22}$$

where $q(z) \in \mathcal{R}^{m}$, $D \in \mathcal{R}^{m\times n}$, Also let the matrix D decomposed as:

$$D = [D_1 \quad I_{m\times m}],\ D_1 \in \mathcal{R}^{m\times(n-m)} \tag{23}$$

When equating the linear map in Eq. (22) to zero, we obtain the required manifold

$$q(z) = [D_1 \quad I_{m\times m}]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = D_1 z_1 + z_2 = 0 \tag{24}$$

or

$$z_2 = -D_1 z_1 \tag{25}$$

The stability of the linear system in Eq. (21-a), can be examined by using Eqs. (21-a) and (25) to get

$$\dot{z}_1 = \tilde{A}_{11} z_1 - \tilde{A}_{12} D_1 z_1 = \left(\tilde{A}_{11} - \tilde{A}_{12} D_1\right) z_1 \tag{26}$$

If $\tilde{A}_{11}$ and $\tilde{A}_{12}$ is a controllable pair then there exist a matrix D_1such that the state will be regulated to the origin ($z_1 = 0$) when starting at the manifold $q(z) = 0$. Consequently, and according to LaSalle invariance principle [9], the sub state z_2 will goes also to zero. This means that the original states x will be regulated to zero, since the transformation T is nonsingular.To this end, the control law u will be determined as follows; we first find the derivative of the manifold functions, Eq. (22), as follows:

$$\dot{q} = DT(Ax + Bu) = DTAx + DTBu \tag{27}$$

Then the following control law will render the manifold ($q(z) = 0$), asymptotically stable

$$u = -(DTB)^{-1}(DTAx + Rq) = -(DTB)^{-1}(DTA + RDT)x \tag{28}$$

provided that DTB is a non singular matrix, i.e.,

$$det(DTB) \neq 0 \tag{29}$$

The above property is insured if and only if the matrix D has a full rank. Note that the matrix R is a diagonal matrix ($R = [r_{ii}]$, $i = 1, 2, \ldots, m$), with r_{ii} as the magnitude of the negative root for the $\dot{q}_i$ dynamics. To verify the stability of the manifold dynamics, we substitute Eq. (28) into Eq. (27)

$$\dot{q} = DTAx + DTB\{-(DTB)^{-1}(DTAx + Rq)\} = -Rq \tag{30}$$

This proves that q decays exponentially to the manifold $q(z) = 0$ with the desired characteristics according to the selected elements for the diagonal matrix R.

Remark 1: the control law in Eq. (28) may also be written in terms of K matrix as

$$u = -Kx \tag{30-a}$$

where $K = (DTB)^{-1}(DTA + RDT) \in \mathcal{R}^{m\times n}$.

Remark 2: the control design based manifold is robust with respect to the parameters uncertainty in A matrix. This can be noticed in Eq. (26) where the design matrix D can be selected such that the uncertain z_1 dynamics is asymptotically sable, and also the whole system via relation (25). This point is clarified in the subsequent section.

Remark 3: one of the direct advantages of designing the manifold based controller is utilized when it is required to confine the state in a certain set in the state space. If the state initiated inside this set, an attractive manifold will arrest the state and bring it to the origin in a prescribed manner.

3. Robustness to the Uncertainty in Matrix A

The effect of the uncertainty in matrix A to the design of the manifold based controller is analysed in this section. The proposed control is robust if it satisfies conditions related to the norm of the uncertainty matrix. To begin, consider the uncertainty in matrix A as follows:

$$\mathrm{A} = \mathrm{A_n} + \Delta\mathrm{A} \tag{31}$$

where $\mathrm{A_n}$ and $\Delta\mathrm{A}$ are the nominal and the uncertainty of matrix A respectively. Here we consider B as a certain matrix, so the transformation matrix T is as given in Eq. (18). Therefore, A is transformed to

$$\begin{aligned}\mathrm{TAT^{-1}} &= \mathrm{T(A_n + \Delta A)T^{-1}} \\ &= \mathrm{TA_nT^{-1} + T\Delta AT^{-1}} = \widetilde{\mathrm{A}}_\mathrm{n} + \Delta\widetilde{\mathrm{A}} \\ &= \begin{bmatrix}\widetilde{\mathrm{A}}_{\mathrm{n}11} & \widetilde{\mathrm{A}}_{\mathrm{n}12} \\ \widetilde{\mathrm{A}}_{\mathrm{n}21} & \widetilde{\mathrm{A}}_{\mathrm{n}22}\end{bmatrix} + \begin{bmatrix}\Delta\widetilde{\mathrm{A}}_{11} & \Delta\widetilde{\mathrm{A}}_{12} \\ \Delta\widetilde{\mathrm{A}}_{21} & \Delta\widetilde{\mathrm{A}}_{22}\end{bmatrix}\end{aligned} \tag{32}$$

The system in Eq. (1) with A as in Eq. (31), is transformed to

$$\dot{\mathrm{z}}_1 = \widetilde{\mathrm{A}}_{\mathrm{n}11}\mathrm{z}_1 + \widetilde{\mathrm{A}}_{\mathrm{n}12}\mathrm{z}_2 + \Delta\widetilde{\mathrm{A}}_{11}\mathrm{z}_1 + \Delta\widetilde{\mathrm{A}}_{12}\mathrm{z}_2 \tag{33}$$

$$\dot{\mathrm{z}}_2 = \widetilde{\mathrm{A}}_{\mathrm{n}21}\mathrm{z}_1 + \widetilde{\mathrm{A}}_{\mathrm{n}22}\mathrm{z}_2 + \Delta\widetilde{\mathrm{A}}_{21}\mathrm{z}_1 + \Delta\widetilde{\mathrm{A}}_{22}\mathrm{z}_2 + \widetilde{\mathrm{B}}\mathrm{u} \tag{34}$$

As in a Backstepping method, z_2 considered as a virtual control. Hence z_2 is used to regulate z_1. The virtual controller is obtained when $\mathrm{q(z)} = 0$, thus $\mathrm{z}_2 = -\mathrm{D}_1\mathrm{z}_1$. Accordingly Eq. (33) becomes:

$$\dot{\mathrm{z}}_1 = (\widetilde{\mathrm{A}}_{\mathrm{n}11} - \widetilde{\mathrm{A}}_{\mathrm{n}12}\mathrm{D}_1)\mathrm{z}_1 + (\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1)\mathrm{z}_1 = (\widetilde{\mathrm{A}}_{\mathrm{n}11} - \widetilde{\mathrm{A}}_{\mathrm{n}12}\mathrm{D}_1)\mathrm{z}_1 + \mathrm{Lz}_1 \tag{35}$$

where

$$\mathrm{L} = \Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1 \tag{36}$$

is the uncertain matrix for z_1 dynamics. The stability of the z_1 dynamics, Eq. (35) may be stated then as follows:

For the case of $\mathrm{L} = 0$, if $\widetilde{\mathrm{A}}_{\mathrm{n}11}$ and $\widetilde{\mathrm{A}}_{\mathrm{n}12}$ is a controllable pair then there exist a matrix D_1 such that the state will be regulated to the origin ($\mathrm{z}_1 = 0$) when it starts at the manifold $\mathrm{q(z)} = 0$. Else if $\mathrm{L} \neq 0$ then sub system in Eq. (35) with matrix D_1 as determined for nominal system parameters must satisfy the following condition:

The matrix M_1 is negative definite for all parameters variation of the matrix $\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1$. The matrix M_1 is given by:

$$\mathrm{M}_1 = P_1(\widetilde{\mathrm{A}}_{11} - \widetilde{\mathrm{A}}_{12}\mathrm{D}_1) + (\widetilde{\mathrm{A}}_{11} - \widetilde{\mathrm{A}}_{12}\mathrm{D}_1)^{\mathrm{T}}P_1$$

or

$$\mathrm{M}_1 = P_1(\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1) + (\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1)^{\mathrm{T}}P_1 - Q_1 \tag{37}$$

where $\tilde{A}_{11} - \tilde{A}_{12}D_1 = (\tilde{A}_{n11} - \tilde{A}_{n12}D_1) + (\Delta\tilde{A}_{11} - \Delta\tilde{A}_{12}D_1)$ and the matrix P_1 is a solution to the Lyapunov equation

$$P_1(\tilde{A}_{n11} - \tilde{A}_{n12}D_1) + (\tilde{A}_{n11} - \tilde{A}_{n12}D_1)^T P_1 = -Q_1 \tag{37-a}$$

for a positive definite matrix Q_1. After that, and according to **LaSalle invariance principle** [9], the sub state z_2 will go also to zero. The next step is the determination of the control law u that will regulate the manifold functions to zero level. Since the matrix A is uncertain, then the control law is derived for $A = A_n$ in a similar way as in Eq. (28).

$$u = -(DTB)^{-1}(DTA_n + RDT)x = -Kx \tag{38}$$

Again, to ensure the asymptotically for the whole system the following condition must be satisfied: The matrix M_2 is negative definite for all parameters variation of the matrix ΔA. The matrix M_2 is given by:

$$M_2 = P_2(A - BK) + (A - BK)^T P_2$$

or

$$M_2 = P_2\Delta A + \Delta A^T P_2 - Q_2 \tag{39}$$

where Q_2 and the matrix P_2 is a solution to the Lyapunov equation for a positive definite matrix

$$Q_2 P_2(A_n - BK) + (A_n - BK)^T P_2 = -Q_2 \tag{39-a}$$

and

$$K = (DTB)^{-1}(DTA_n + RDT) \tag{39-b}$$

4. Manifold Based Controller (MBC) Design Procedure

Eventually, and by considering the control law derived in the previous section, the manifold based controller design procedure can be summarized by the following steps:

- Compute the transformation matrix T according to Eq. (18).
- Determine the sub matrices $\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{21}$, and $\tilde{A}_{22}$ from the following relation with the dimensions specified above

$$TAT^{-1} = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

- The selection of the matrix D is according to the desired characteristic roots for the z_1 subsystem in Eq. (26) ($\tilde{A}_{11}$ and $\tilde{A}_{12}$ must be a controllable pair). In addition the matrix *D* must have full rank to ensure the invertability of the matrix DTB.
- The attractiveness of the manifold $q(z) = 0$ is ensured via selection of a positive elements for the matrix R.
- Finally the control law is;

$$u = -Kx \tag{40}$$

where,

$$K = (DTB)^{-1}(DTA + RDT) \tag{41}$$

is $m \times n$ matrix.

- If the matrix A is uncertain, Eq. (31), the control law in Eq. (40) is designed as in steps 1 to 5, with the $\mathrm{A} = \mathrm{A_n}$. The whole system is then asymptotically stable provide that the inequalities (37) and (39) are satisfied.

In the subsequent section, the above presented procedure will be used to the control design for a linearized model for the crane system.

5. Crane System Dynamics

The general definition of a crane is a mechanical system designed to lift and move loads through a hook suspended from a movable arm or trolley. Safety and economic constraints require that both the load swing and the transfer time are kept as small as possible.

For the system in Fig. 1, a generalized coordinate q can be taken as $\mathrm{q} = [\theta\ \mathrm{x}]^{\mathrm{T}}$. Then the kinetic energy function $T$and the potential energy function $V$are given as follows:

$$T = T_{trolley} + T_{load}$$
$$= \frac{1}{2}m_T\dot{\mathrm{x}}^2 + \frac{1}{2}m_L(\dot{\mathrm{x}}_\mathrm{L}^2 + \dot{\mathrm{y}}_\mathrm{L}^2)$$
$$= \frac{1}{2}m_T\dot{\mathrm{x}}^2 + \frac{1}{2}m_L\left(\dot{\mathrm{x}}^2 + 2\dot{\mathrm{x}}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) \tag{42}$$

$$V = m_L gl\cos\theta\,(1 - \cos\theta) \tag{43}$$

where m_T and m_L are the trolley and the load masses respectively, l is the rope length, x_L and y_L: are the load Cartesian coordinates respectively.

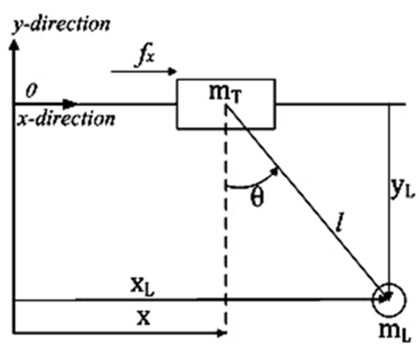

Fig. 1. An Equivalent Diagram for the Suspended Load on the Electro-Mechanical System.

By constructing the system Lagrangian $L = T - V$ and using Euler-Lagrange's equation defined as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1,2 \tag{44}$$

where the generalized input Q is $\left[0\, f_x\right]^T$, then we obtain the following equations of motion:

$$(m_T + m_L)\ddot{x} + m_L l\ddot{\theta}\cos(\theta) - m_L\dot{\theta}^2\sin(\theta) = f_x \tag{45}$$

$$\ddot{x}\cos(\theta) + l\ddot{\theta} + g\sin(\theta) = 0 \tag{46}$$

Here f_x is the horizontal force acting on the trolley. Since f_x is generated by a DC motor which is operated due to an electrical voltage u then f_x is written as

$$f_x = k_u u \tag{47}$$

where k_u is the DC motor's constant. Substitute for f_x (Eq. (47)) and rearrange Eqs. (45) and (46) result in the following form

$$\ddot{x} = \frac{1}{\left(m_T + m_L(1 - cos(\theta)^2)\right)}\left[m_L g\, sin(\theta)\, cos(\theta) + m_L\dot{\theta}^2\, sin(\theta) + k_u u\right] \tag{48}$$

$$\ddot{\theta} = \frac{1}{l}\left[-\left[\frac{1}{\left(m_T + m_L(1 - cos(\theta)^2)\right)}\left[m_L g\, sin(\theta)\, cos(\theta) + m_L\dot{\theta}^2\, sin(\theta) + k_u u\right]\right] cos(\theta) - g\, sin(\theta)\right] \tag{49}$$

For a small angle θ the linearized model of the crane system is given by

$$\ddot{\theta} = -\frac{(m_T + m_L)g}{m_T l}\theta - \frac{k_u}{m_T l}\, u \tag{50}$$

$$\ddot{x} = \left[\frac{m_T g + m_L g - m_t g}{m_T}\right]\theta + \frac{k_u}{m_t}\, u \tag{51}$$

In state-space form, the following states are defined by:

$x = x_1$, $\dot{x} = x_2$, $\theta = x_3$, and $\dot{\theta} = x_4$

Now Eq. (50) and (51) in state-space variables are:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{m_L g}{m_T}x_3 + \frac{k_u}{m_T}\, u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\frac{(m_T + m_L)g}{m_T l}x_3 - \frac{k_u}{m_T l}\, u \end{aligned}\right\} \tag{52}$$

Or in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \frac{m_L}{m_T}g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{(m_T + m_L)g}{m_T l} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{k_u}{m_T}\\ 0\\ \frac{-k_u}{m_T l}\end{bmatrix} u = Ax + Bu \tag{53}$$

The final linearized model will be used in the subsequent sections to design the manifold based controller.

6. MBC Design for Container Crane System

In order to perform controller gain calculation, the simulation parameters given below in Table 1 are taken from an experimental pendulum system built by Feedback Instruments Ltd. [10].

Table 1. Crane System Model Parameters.

Parameter	Value	Units
Trolley Mass m_T	2.4	kg
Load Mass m_L	0.23	kg
Rope length l	0.4	m
DC motor constant k_u	8	N/volt
Desired trolley position x_d	1	m

By utilizing the above parameters and according to the controller design procedure we ever mention, we first calculate the transformation matrix T, which is found to be:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & -0.4 & 0 & 1 \end{bmatrix} \tag{54}$$

After that we need to calculate the transformed plant matrix $\tilde{A}$:

$$\tilde{A} = \begin{bmatrix} 0 & 0.8621 & 0 & -0.3448 \\ 0 & 0 & -9.8 & 0 \\ 0 & 0.3448 & 0 & 0.8621 \\ 0 & 0 & -27.2236 & 0 \end{bmatrix} \tag{55}$$

where:

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0.8621 & 0 \\ 0 & 0 & -9.8 \\ 0 & 0.3448 & 0 \end{bmatrix}, \tilde{A}_{12} = \begin{bmatrix} -0.3448 \\ 0 \\ 0.8621 \end{bmatrix}$$

$\tilde{A}_{11} = [0 \quad 0 \quad -27.2236]$, and $\tilde{A}_{22} = 0$

In order to calculate the matrix D, it is needed first to choose the roots of the desired characteristics equation that will placed for the z_1 subsystem in Eq. (26). Notes that $\tilde{A}_{11}$ and $\tilde{A}_{12}$should be a controllable pair which can be easily verified by calculating the controllability matrix of this pair. After calculation this pair is found to be fully controllable. The desired characteristics roots are chosen to be:

$$Desired\ roots = [-2 \quad -3 \quad -4] \tag{56}$$

Applying pole placement to find the matrix D_1, this resulted in a matrix D to be as:

$$D = [-2.8408 \quad -2.6776 \quad 9.3037 \quad 1] \tag{57}$$

In order to ensure that the $q(z)$ function decays exponentially to the manifold $q(z) = 0$, we should choose matrix R which for our case is scalar value and is taken as $\mathrm{R} = \mathrm{I}$, where I is the identity matrix. Finally, the gain matrix K is calculated as follows (Eq. (41)):

$$K = [0.2939 \quad 0.6122 \quad -0.8607 \quad -0.9551] \tag{58}$$

The closed loop poles for the closed loop matrix $A - BK$ are

$$closed\ loop\ roots = [-4 \quad -3 \quad -2 \quad -1]$$

This proves the exponential asymptotical stability for the closed loop system. When the load mass is variable; the matrix A, from the control theory point of view, is considered uncertain. To take into account the uncertainty in matrix A, we first take $\mathrm{A_n}$ equal to A, Eq. (53) and then calculate the control law by following the steps presented above. Hence the matrix K is as given in Eq. (58).With this control law the stability will not be ensured unless the negative definite condition imposed on the matrices (37) and (39) are satisfied. As a converse idea, the band of uncertainty in the load mass could be estimated using the definiteness condition stated above. Our calculation in *Appendix A*, showed that the matrix $\mathrm{M_1}$ is negative definite and does not affected by the variation of the load mass, while for matrix $\mathrm{M_2}$ it was found negative definite for the following band: -0.064 kg $< \Delta m < 1.283$ kg

7. Simulation Results

By utilizing the values presented in Table 1 to calculate the controller gain K, Eq. (41) where it's values found to be as stated in Eq. (58).The simulation results are performed using Matlab/Simulink (ver.14.9-2009b) and the designed Simulink model is shown in Fig. 2, where the simulation results presented here is achieved using the nonlinear crane system model Eqs. (48) and (49).

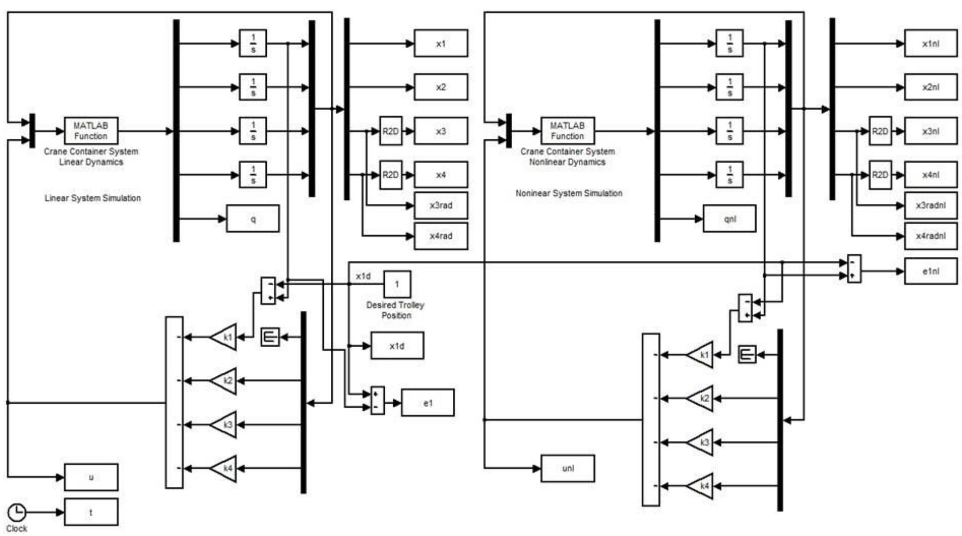

Fig. 2. Matlab/Simulink Crane System Model.

The simulation first is performed by taking the nominal value of load mass $m_L$in Table 1. Figures 3 to 6 show the system variables and performance when the controller we designed in Section 6 is applied to the crane system.

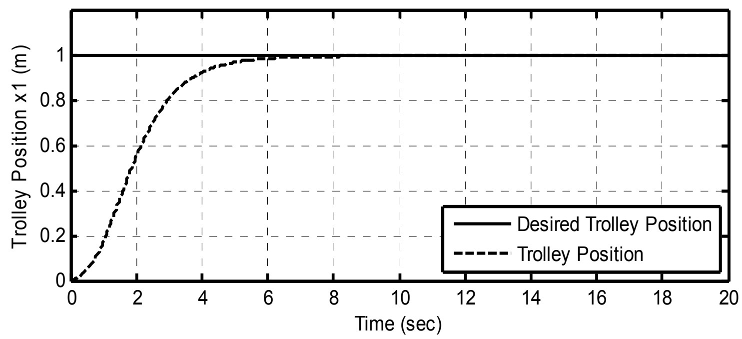

Fig. 3. Trolley Position Time History Simulation Results for Nominal Load Mass m_L Value.

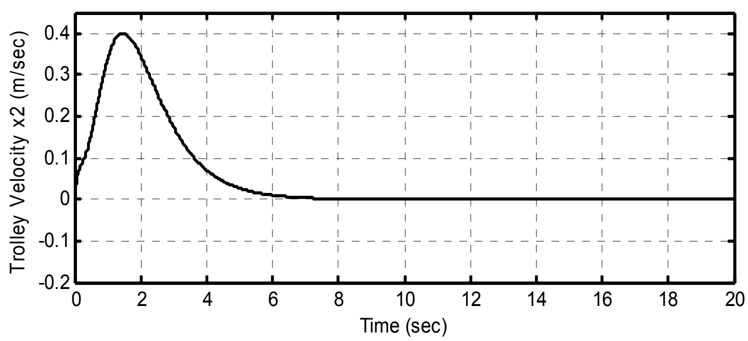

Fig. 4. Trolley Velocity Time History Simulation Results for Nominal Load Mass m_L Value.

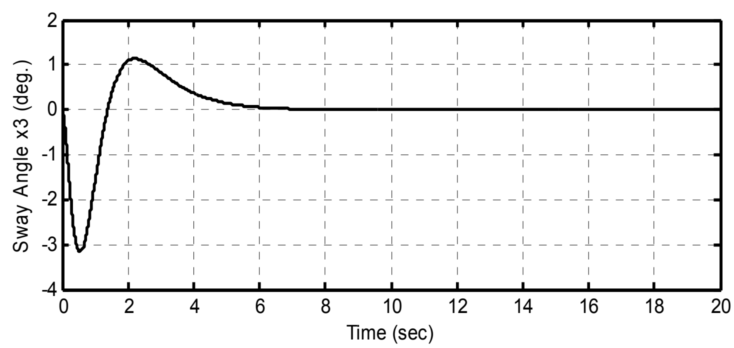

Fig. 5. Sway Angle Time History Simulation Results for Nominal Load Mass m_L Value.

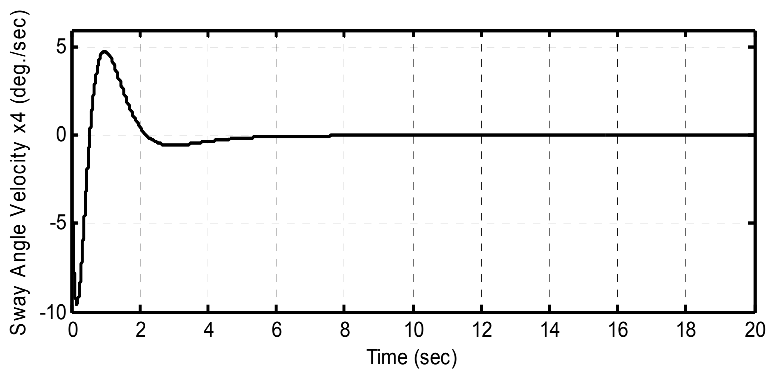

Fig. 6. Sway Angle Velocity Time History Simulation Results for Nominal Load Mass m_L Value.

Control voltage versus time and the phase plot for sway angle vs. sway angle velocity are displayed in Figs. 7 and 8 respectively, while manifold function $q(z)$ time history is shown in Fig. 9. It can be seen clearly that Trolley position $x_1$that is shown in Fig. 4 is approaching the desired distance in about 6 s. while the swing angle do not exceed ± 4 deg. as shown in Fig. 5. The controller signal succeeds to bring the manifold function $q(z)$ to zero as shown in Fig. 9 in about 4 s. In order to validate our controller design to the uncertainty in A matrix, the suggested controller is simulated again by perturbing the load mass m_L value and taking it as double value 0.46. For this case the poles of the closed system are;

$$closed\ loop\ roots = [-3.9914 \pm 2.0293i \quad -1.0085 \pm 0.4243i]$$

The simulation results show the potential of the designed controller because it succeeds to achieve the objectives of the control problem in the same way as in first case (nominal load mass m_L). Figures 10 to 13 show the system variables and performance for the second case simulation.

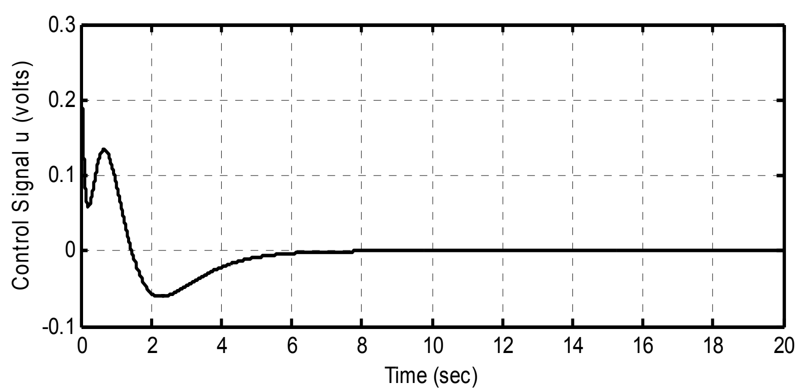

Fig. 7. Control Voltage Time History Simulation Results for Nominal Load Mass m_L Value.

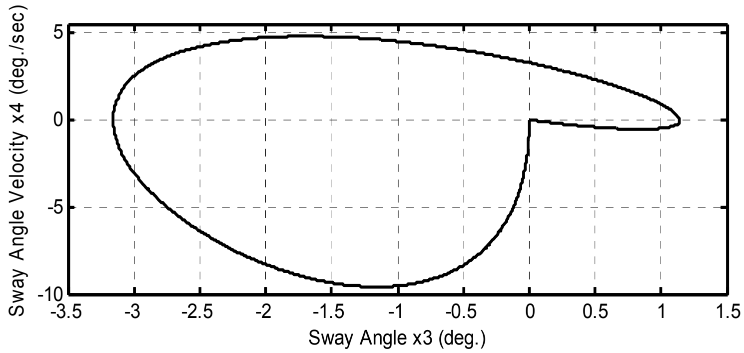

Fig. 8. Phase Plot of Sway Angle vs. Sway Angle Velocity Simulation Results for Nominal Load Mass m_L Value.

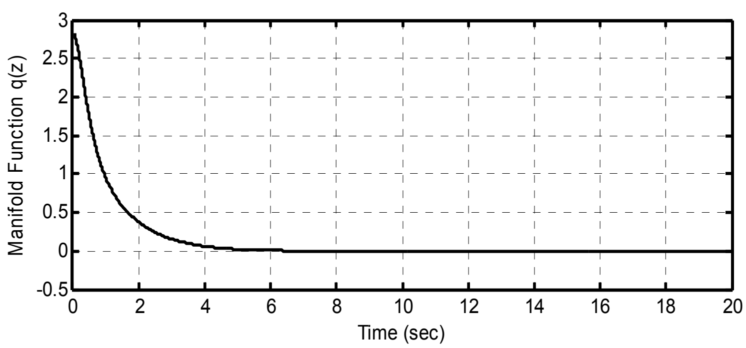

Fig. 9. Manifold Function $q(z)$ Time History Simulation Results for Nominal Load Mass m_L Value.

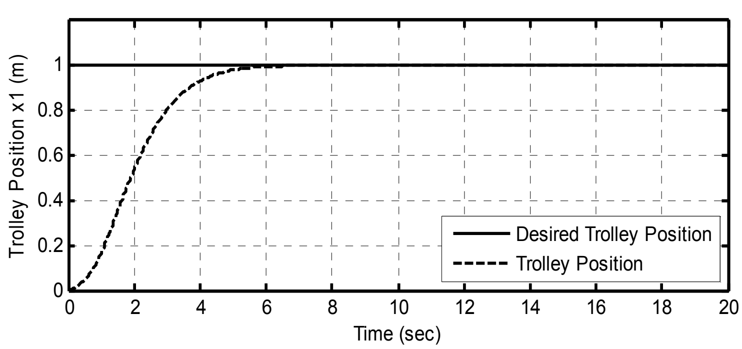

Fig. 10. Trolley Position Time History Simulation Results for Perturbed Load Mass m_L Value.

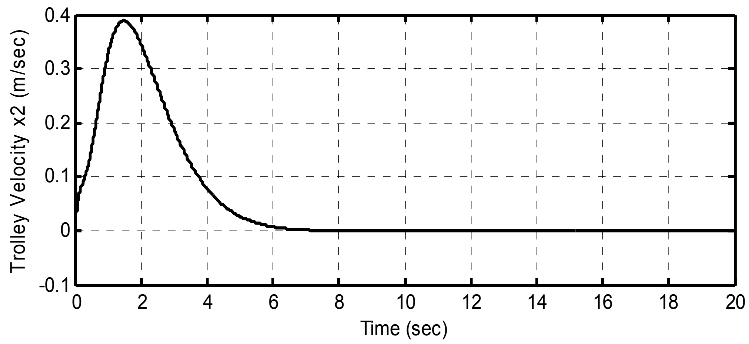

Fig. 11. Trolley Velocity Time History Simulation Results for Perturbed Load Mass m_L Value.

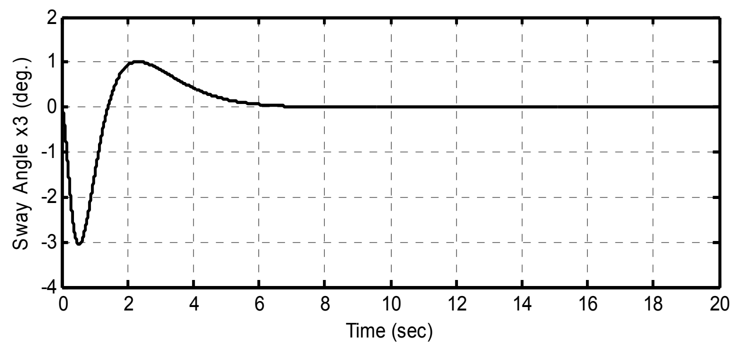

Fig. 12. Sway Angle Time History Simulation Results for Perturbed Load Mass m_L Value.

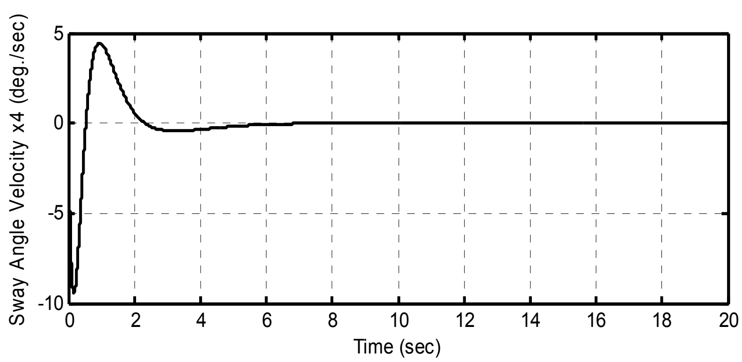

Fig. 13. Sway Angle Velocity Time History Simulation Results for Perturbed Load Mass m_L Value.

Control voltage versus time and the phase plot for sway angle vs. sway angle velocity are displayed in Figs. 14 and 15 respectively, while manifold function $q(z)$ time history is shown in Fig. 16. The manifold based controller enforce the trolley position x_1 that is shown in Fig. 10 to track the desired distance in about 6 s. while the swing angle do not exceed ± 4 deg. as shown in Fig. 12. The main feature here is that the controller signal also brings the manifold function $q(z)$ to zero as shown in Fig. 16 in about 4 s also.

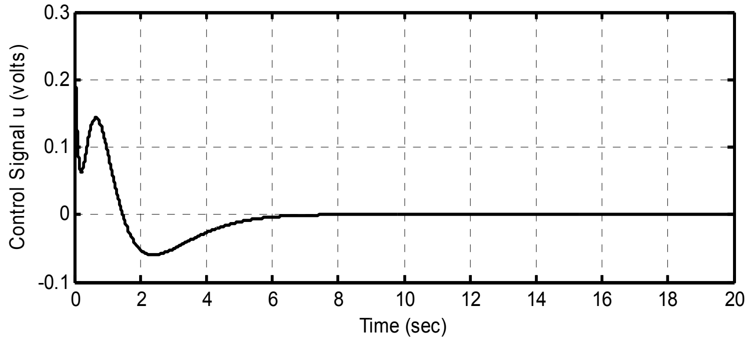

Fig. 14. Control Voltage Time History Simulation Results for Perturbed Load Mass m_L Value.

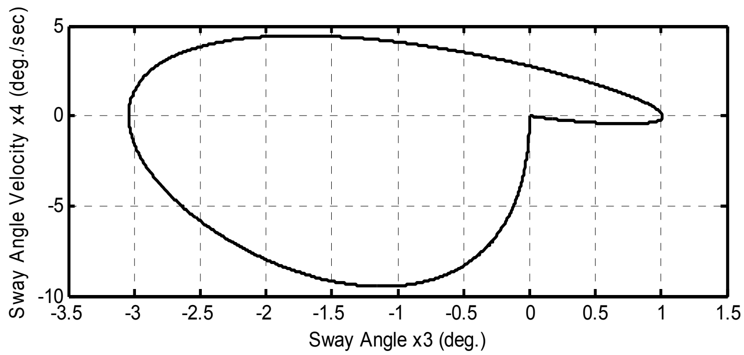

Fig. 15. Phase Plot of Sway Angle vs. Sway Angle Velocity Simulation Results for Perturbed Load Mass m_L Value.

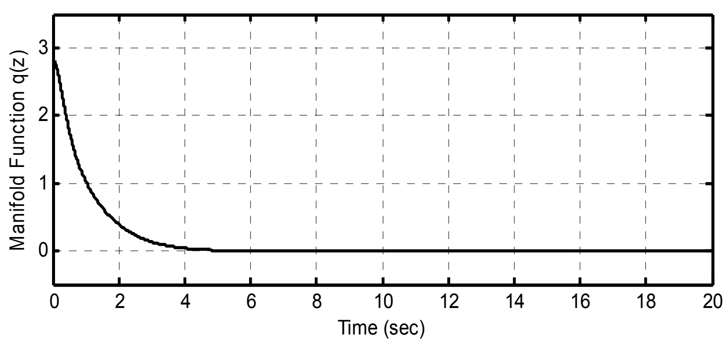

Fig. 16. Manifold Function $q(z)$ Time History Simulation Results for Perturbed Load Mass m_L Value.

8. Conclusions

In this paper a linear state feedback controller has been derived based on the creation of an outputs named as manifold functions. The manifold function was determined in such a way that the upper subsystem of the regular form is asymptotically stable. The function of the controller is then to regulate $q(z)$ manifold function asymptotically to zero level; consequently the whole system is stabilized. The result is a new linear feedback control structure based on the manifold function. The manifold based control has been designed for a container crane system and the results showed the ability of the controller in translating the load to a desired position for a time period less than 8 seconds where the sway angle not exceeding ±4 degree. Eventually, the simulation results show that manifold based control is robust when the load mass is doubled. In fact the robustness of the manifold based control could be enhanced by careful design of the manifold functions.

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Appendix A

Calculating the Load Mass Variation Band

The estimation for the load mass of the container crane such that the stability is ensured in spite of using the control law based on the nominal parameters is presented in this appendix. The calculation steps are as follows:

Step I: for $Q_1 = I_{3\times3}$ and $Q_2 = I_{4\times4}$ calculate P_1 and P_2 according to Lyapunov equations in (37-a) and (39-a), where $I_{3\times3}$ and $I_{4\times4}$ are 3×3 and 4×4 identity matrices respectively.

Step II: constructing the matrices M_1 and M_2 according to Eqs. (37) and (39) respectively.

Step III: by using the MATLAB symbolic determinate, we examine the negative definiteness for M_1 and $\mathrm{M}_2$and from which we determine the variation band for the load mass.

For M_1 we apply the above steps and the following results are found:

$$\Delta\widetilde{\mathrm{A}}_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Delta\widetilde{\mathrm{A}}_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{M}_1 = P_1\left(\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1\right) + \left(\Delta\widetilde{\mathrm{A}}_{11} - \Delta\widetilde{\mathrm{A}}_{12}\mathrm{D}_1\right)^{\mathrm{T}} P_1 - \mathrm{I}_{3\times3} = -\mathrm{I}_{3\times3}$$

This proves that M_1 is negative definite irrespective to the variation in the load mass. This situation, from the other hand, is because the uncertainty in matrix A, due to load mass variation, satisfying the matching condition.

For $\mathrm{M}_2$the results for the calculations are:

$$A_n - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.9797 & -2.0407 & 3.8082 & 3.1837 \\ 0 & 0 & 0 & 1 \\ 2.4492 & 5.1017 & -34.0204 & -7.9592 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1.7090 & 0.8905 & -1.6980 & 0.1520 \\ 0.8905 & 1.1515 & -2.4737 & 0.1880 \\ -1.6980 & -2.4737 & 8.6479 & -0.2622 \\ 0.1520 & 0.1880 & -0.2622 & 0.1051 \end{bmatrix}$$

$$\Delta\mathrm{A} = \left(\frac{9.8}{2.4}\right)\Delta\mathrm{m_L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 0 \end{bmatrix}$$

$$\mathrm{M}_2 = P_2\Delta\mathrm{A} + \Delta\mathrm{A}^{\mathrm{T}}P_2 - \mathrm{I}_{4\times4}$$

$$= \begin{bmatrix} -1 & 0 & 2.0840 * \Delta\mathrm{m_L} & 0 \\ 0 & -1 & 2.7823 * \Delta\mathrm{m_L} & 0 \\ 2.0840 * \Delta\mathrm{m_L} & 2.7823 * \Delta\mathrm{m_L} & -14.8488 * \Delta\mathrm{m_L} - 1 & -0.3050 \\ 0 & 0 & -0.3050 & -1 \end{bmatrix}$$

The matrix M_2 is negative definite if the determinants of the leading principal minors is alternating in sign, i.e.,

$$\mathrm{M}_{11} = -1 < 0,\ \mathrm{M}_{22} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 > 0$$

$$\mathrm{M}_{33} = \begin{vmatrix} -1 & 0 & 2.0840 * \Delta\mathrm{m_L} \\ 0 & -1 & 2.7823 * \Delta\mathrm{m_L} \\ 2.0840 * \Delta\mathrm{m_L} & 2.7823 * \Delta\mathrm{m_L} & -14.8488 * \Delta\mathrm{m_L} - 1 \end{vmatrix} < 0 \ for\ -0.064\ \mathrm{kg} < \Delta m < 1.283\ \mathrm{kg}$$

$$\mathrm{M}_{33} = |\mathrm{M}_2| > 0\ for\ -0.064\ \mathrm{kg} < \Delta m < 1.283\ \mathrm{kg}$$

These calculations are carried out using MATLAB symbolic determinate and equation solver to determine the band Δm.