

APPROXIMATION OF FREE-FORM CURVE – AIRFOIL SHAPE

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Abstract

Approximation of free-form shape is essential in numerous engineering applications, particularly in automotive and aircraft industries. Commercial CAD software for the approximation of free-form shape is based almost exclusively on parametric polynomial and rational parametric polynomial. The parametric curve is defined by vector function of one independent variable $\mathbf{R}(u) = (x(u), y(u), z(u))$, where $0 \leq u \leq 1$. Bézier representation is one of the parametric functions, which is widely used in the approximating of free-form shape. Given a string of points with the assumption of sufficiently dense to characterise airfoil shape, it is desirable to approximate the shape with Bézier representation. The expectation is that the representation function is close to the shape within an acceptable working tolerance. In this paper, the aim is to explore the use of manual and automated methods for approximating section curve of airfoil with Bézier representation.

Keywords: Shape approximation, Parametric polynomial, Bézier representation, Free-form shape.

1. Introduction

Before the advent of the computer, section of curves had been constructed by full-scale manual drawing, particularly for air-craft industry during Second World War. Constructing the full-scale drawing is a time consuming task, even for highly skilled draughtsmen. In addition, storage of full-scale manual drawing is practically inconvenient.

With the advent of computer, many of mathematical techniques were developed, mostly departing from the concept of manual drawing [1]. The fundamental idea is to develop parametric polynomial to represent the curve shape which is defined by vector function of one independent variable $\mathbf{R}(u)$, where

Nomenclatures

$B_i^n(u)$	Bernstein polynomial
d	Maximum shortest distance, m
f	Minimising function
k_0, k_3	Scaling factors
\mathbf{P}_i	Data Points
$\mathbf{R}(u)$	Parametric function
\mathbf{V}_i	Control points

$\mathbf{R}(u) = (x(u), y(u), z(u))$, where $0 \leq u \leq 1$. One of the parametric polynomials - Bézier representation, has been widely used to address curve shape approximation problem [2-5].

There is a number of research has been carried out in airfoil design by using Bézier representation. For designing airfoil shape, Bézier representation has been used to design a medial curve for the construction of entire airfoil shape [6]. Bézier representation offers simple interactive shape modification and computationally efficient interrogations [7], which has been used to replace the existing mathematical formulation for the design of airfoil shape based on prescribe aerodynamics requirement [8]. Furthermore, an aerodynamic investigation shows a promising result for the airfoil shape is designed by Bézier representation [9]. Therefore, Bézier representation plays a significant role in the design of airfoil, where geometric shape dominates the overall performance [10-12].

It is noted that the above mentioned literatures focused on aerodynamic performance of airfoil shape. In this paper, the attention is turning to the geometric aspect of airfoil shape. The starting point of the research is based on a string of points with the assumptions of sufficiently dense to characterise airfoil shape as shown in Fig. 1. The aim of this paper is to use manual and automated methods to approximate the airfoil shape with Bézier representation. The expectation is that the representation function is close to the shape within an acceptable tolerance. It is questionable that what should be the acceptable tolerance. In this paper, the acceptable tolerance is based on machining practice, which is within 0.1 mm [13]. Apart from satisfying the tolerance, the smooth transition of the curve shape is desirable.

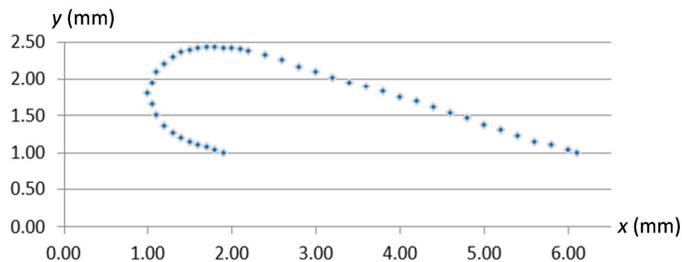


Fig. 1. Data Points that Characterising Airfoil Shape.

The remaining sections of this paper are organized as follows. Section 2 introduces the Bézier curve representation and highlights its properties. Section 3

explains the manual method of approximating the airfoil shape and discusses the significance of graphical results generated by manual method. Section 4 presents the underlying mathematics for the automated method to approximate the airfoil shape, and then the corresponding graphical results will be discussed. Section 5 summarise the overall achievement of the research and reviews critically the performance of the approximation methods.

2. Definition and Properties of Bézier Curve

Bézier representation is developed by a mechanical engineer from Renault Car Company in seventies [14]. The underlying idea is to formulate the parametric polynomial in such a way that the shape control parameters are explicitly defined by control points which are not necessarily lie on the curve. A parametric Bézier curve of degree n is defined as follow:

$$\mathbf{R}(u) = \sum_{i=0}^n B_i^n(u) \mathbf{V}_i, \quad 0 \leq u \leq 1 \quad (1)$$

where \mathbf{V}_i are the control points and $B_i^n(u)$ is the Bernstein polynomial.

$$B_i^n(u) = {}_n C_i u^i (1-u)^{n-i}, \quad 0 \leq i \leq n$$

In this paper, the cubic Bézier curve ($n = 3$) will first be used, which is the lowest degree of Bézier curve that allows the control of end tangent vectors. The cubic Bézier curve is expressed as follow:

$$\mathbf{R}(u) = (1-u)^3 \mathbf{V}_0 + 3u(1-u)^2 \mathbf{V}_1 + 3u^2(1-u) \mathbf{V}_2 + u^3 \mathbf{V}_3 \quad (2)$$

The example of cubic Bézier curve is shown in Fig. 2. Cubic Bézier curves have many important properties that are useful for the approximation of airfoil shape, which are given as follows:

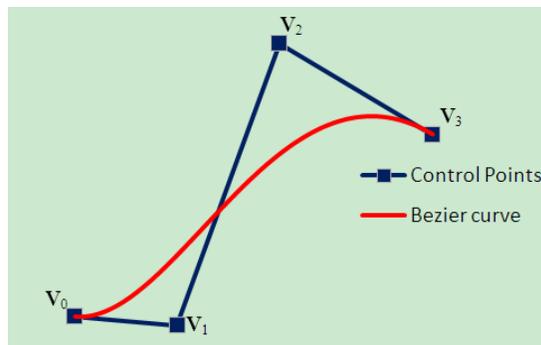


Fig. 2. Cubic Bézier Curve with Control Points.

- \mathbf{V}_0 and \mathbf{V}_3 are the two endpoints of the curve segment. This can be proved when substituting $u = 0$ and $u = 1$ respectively in Eq. (2).
- The convex hull property confines the curve to be lied within the polygon formed by the control points \mathbf{V}_0 , \mathbf{V}_1 , \mathbf{V}_2 and \mathbf{V}_3 , which provides clue for the size limit of the curve.

- \mathbf{V}_1 and \mathbf{V}_2 are the two inner control points (see Fig. 2) and are located on the tangent vectors on the curve at \mathbf{V}_0 and \mathbf{V}_3 , respectively. That is if k_0 and k_3 are scalars, we have

$$\mathbf{V}_1 = \mathbf{V}_0 + \Delta\mathbf{V}_0 \times k_0$$

$$\mathbf{V}_2 = \mathbf{V}_3 + \Delta\mathbf{V}_3 \times k_3$$

Subsequently, the quintic Bézier curve ($n=5$) will be used, which is expressed as follow:

$$\mathbf{R}(u) = (1-u)^5\mathbf{V}_0 + 5u(1-u)^4\mathbf{V}_1 + 10u^2(1-u)^3\mathbf{V}_2 + 10u^3(1-u)^2\mathbf{V}_3 + 5u^4(1-u)\mathbf{V}_4 + u^5\mathbf{V}_5 \quad (3)$$

Similar to cubic Bézier curve, the quintic Bézier curve possess the 3 properties as highlighted before. In particular, the tangent control is given by inner control points \mathbf{V}_1 and \mathbf{V}_4 . In addition to that, the extra control of second derivative by inner control points \mathbf{V}_2 and \mathbf{V}_3 is provided. The procedures to approximate airfoil shape by using cubic and quintic Bézier curves will be described in Section 3.

3. Manual Method of Airfoil Shape Approximation

In this paper, the cubic Bézier curve ($n=3$) is firstly considered, which is essentially the lowest degree of curve that allows tangent control for two end points. For quartic Bézier curve ($n=4$), the extra control point of \mathbf{V}_2 allows second derivative control for two end points. In this sense, second derivative of two different parameters (two end points) is controlled simultaneously by only one parameter \mathbf{V}_2 . This unbalance controlled on second derivative is undesirable, even though quartic Bézier curve is higher degree than cubic Bézier curve.

For quintic Bézier curve ($n=5$), there are two extra control points of \mathbf{V}_2 and \mathbf{V}_3 . This allows full control of second derivative for planar curve, where the curve shape can be fully characterized as can be traced back to the theory of differential geometry [15]. Consequently, it is not necessary to consider higher degree Bézier curve. The focus is only on cubic and quintic Bézier curves are considered in this paper. The procedures to approximate the airfoil shape characterized by a string of points are shown in Fig. 3.

Given a set of points P_i ($i = 0, 1, 2, 3, \dots, N$), the aim is to apply a Bézier curve ($n = 3, 5$) that is able to approximate the set of points within an acceptable tolerance. Initially the end control points are adjusted to be coincided with the end data points. Subsequently, the internal control points $\mathbf{V}_0^k, \dots, \mathbf{V}_{n-1}^k$ are manually adjusted and then the respective Bézier curve $\mathbf{R}^k(u)$ can be calculated, where k is the number of iteration. The role of internal control points for both the cubic Bézier Curve and quintic Bézier Curve have been described in Section 2.

Once the control points are adjusted, the discrepancies between the data points and Bézier curve are determined based on shortest distance. It is noted that the shortest distance is invariant under geometric transformation, i.e., translation and rotation. If the maximum shortest distance d is within tolerance of 0.1 mm, the Bézier curve $\mathbf{R}(u)$ can then be finalized; otherwise the internal control points need to be readjusted.

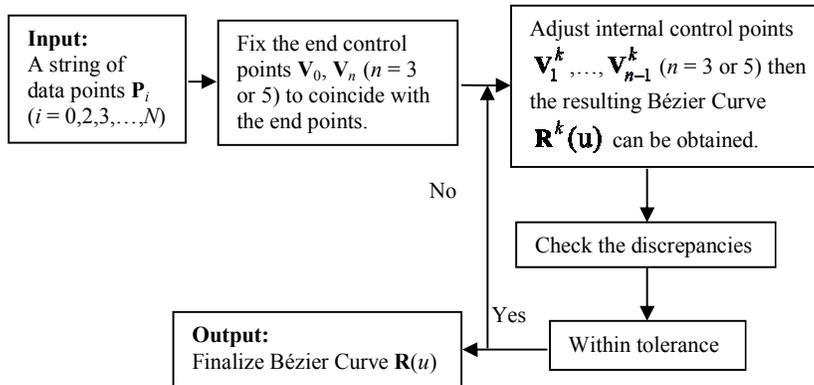


Fig. 3. Procedures of Approximating Airfoil Shape.

The method of determining maximum shortest distance d can be found in [16]. For determining the maximum shortest distance d , the first step is to discretize the curve $R^k(u)$ into a set of points $R^k(u_i)$, where $i = 0, 1, 2, \dots, m$. The maximum shortest distance is defined in such a way that $d = \max \left\{ \min \left\| P_i - R^k(u_i) \right\| \right\}$. For the point P_0 , the idea is to use branch-and-bound search principle from P_0 to the neighboring point set $R^k(u_i)$ until the shortest distance $\min \left\| P_i - R^k(u_i) \right\|$ is obtained. For the remaining of the points P_i , the similar method to determine shortest distance is used. Once all the shortest distances from P_i to $R^k(u_i)$ are determined, the maximum shortest distance d is essentially the greatest value among all the shortest distances.

In the following subsections, the manual method will be applied to approximate the airfoil shape by using cubic and quintic Bézier Curves. Subsequently, the generated graphical results will be discussed.

3.1. Airfoil shape approximation by cubic Bézier curve

This subsection discusses the approximation results of air-foil by cubic Bézier curve. The resulting control points are given by $V_0(1.9, 1)$, $V_1(0.8, 1.1)$, $V_2(-0.6, 4)$, $V_3(6.1, 1)$ as shown in Fig. 4.

The maximum error of 0.093 mm occurs at the flatter region of the airfoil as shown in Fig. 5, which is within the acceptable tolerance of 0.1 mm. The challenging part of the approximation is at the leading edge (see Fig. 5) of the airfoil shape, which mainly governs by control points V_1 and V_2 . The control point V_1 deals with the tangent of first point (see Fig. 5), in which the bending direction of the curve is defined.

As mentioned in Section 2, control point V_2 supposes to be mainly governs the tangent at the last point as shown in Fig. 5. However, due to the sharp bend feature of leading edge, the flatter region of the airfoil shape need to be compromised for capturing the shape of leading edge region. In this sense, the V_2

is simultaneously controlling the tangent at last point and capturing the shape of leading edge region. This cause the miss-capture of the leading edge region and the undesirable sudden transition is formed as shown in Fig. 5.

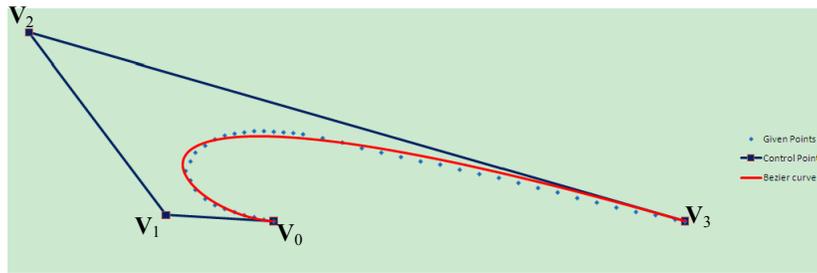


Fig. 4. Resulting Cubic Bézier Curve that Approximating Airfoil Shape.

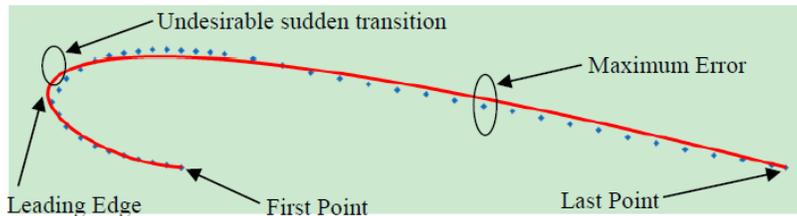


Fig. 5. Enlargement View of the Approximating Airfoil Shape.

From the approximation of airfoil shape with cubic Bézier curve, it is suggested that the curve do not have sufficient degree of freedom to capture the airfoil shape even though the maximum error of 0.093 mm is within an acceptable tolerance. Therefore, the quintic Bézier curve will be considered in subsection 3.2, which provides higher degree of freedom in shape control.

3.2. Airfoil shape approximation by quintic Bézier curve

This subsection discusses the approximation results of air-foil by quintic Bézier curve. The resulting control points are given by $V_0(1.9,1)$, $V_1(0.6,1.3)$, $V_2(0.9,2.2)$, $V_3(1, 3.2)$, $V_4(2.4, 3.2)$, $V_5(6.1,1)$ as shown in Fig. 6.

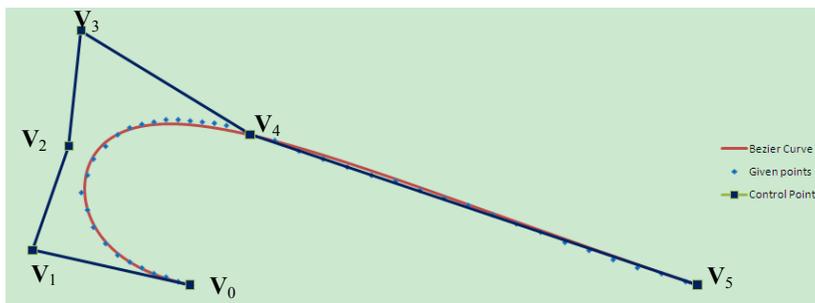


Fig. 6. Resulting Quintic Bézier Curve that Approximating Airfoil Shape.

The maximum error of the approximation is 0.037 mm at the highly curved region as shown in Fig. 7, which has been significantly reduced if compared to the cubic Bézier curve approximation as discussed in subsection 3.1. In addition, the control point V_2 provides the control of second derivative, which enables the curve shape to have smooth transition as shown in Fig. 7.

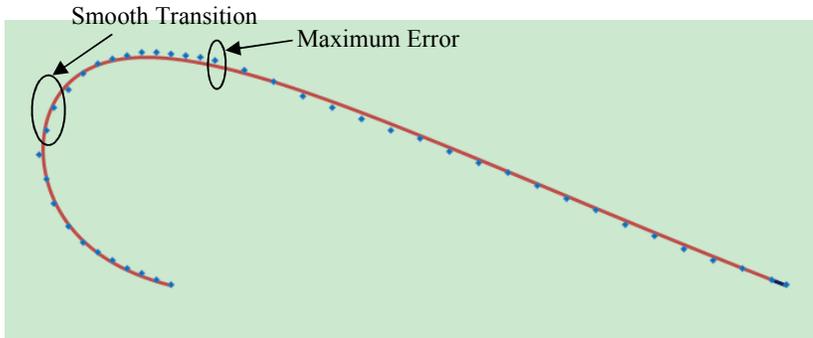


Fig. 7. Enlargement View of the Approximating Airfoil Shape.

The control point V_3 is able to concentrate on capturing the shape of highly curved region. Whereas, the control point V_4 is able to concentrate on capturing the flatter region. In this sense, the control points V_3 and V_4 address the problem of shape miss-capturing as discussed in Section 4. In short, the quintic Bézier curve has higher degree of freedom in shape capturing, which produce better shape approximation than cubic Bézier curve. Therefore, it could be worthwhile to explore the use of quintic Bézier curve for automated approximation of airfoil shape. The detail of the automated approximation will be presented in Section 4.

4. Automated Method for Airfoil Shape Approximation

The aim of this section is to present an underlying mathematics for automated approximation of airfoil shape, which avoids the steps of manual iteration on control points as described in Section 3. Considering a set of points P_i ($i = 0, 1, 2, 3, \dots, N$), the idea is to use least squares approximation [17] so that the control points V_i ($0, 1, 2, \dots, 5$) of quintic Bézier curve $R(u)$ can be solved linearly.

As of (3.2), the quintic Bézier curve can be written in the form of

$$\mathbf{R}(u) = \sum_{i=0}^5 B_i^5(u) \mathbf{V}_i, \quad 0 \leq u \leq 1 \quad (4)$$

Satisfying that:

- $\mathbf{P}_0 = \mathbf{R}(0)$ and $\mathbf{P}_N = \mathbf{R}(1)$
- The remaining \mathbf{P}_k are approximated in the least square sense, i.e.,

$$f = \sum_{k=1}^{N-1} \|\mathbf{P}_k - \mathbf{R}(\bar{u}_k)\|^2 \quad (5)$$

is a minimum with respect to the variables $\mathbf{V}_i (i = 1, \dots, 4)$, the \bar{u}_k are parameter values.

For the sake of convenient in mathematics manipulation, let

$$\mathbf{Q}_k = \mathbf{P}_k - B_0^5(\bar{u}_k)\mathbf{P}_0 - B_5^5(\bar{u}_k)\mathbf{P}_N \quad k = 1, \dots, N-1 \tag{6}$$

By substituting Eqs. (4) and (6) into Eq. (5), it can be obtained that

$$f = \sum_{k=1}^{N-1} \left\| \mathbf{Q}_k - \sum_{i=1}^4 B_i^5(\bar{u}_k)\mathbf{V}_i \right\|^2 \tag{7}$$

For least square approximation, the idea is to minimize f .

Let $\frac{\partial f}{\partial \mathbf{V}_l} = 0$, where $l = 1, \dots, 4$, it can then obtain

$$\frac{\partial f}{\partial \mathbf{V}_l} = \sum_{k=1}^{N-1} \left(-2B_l^5(\bar{u}_k)\mathbf{Q}_k + 2B_l^5(\bar{u}_k) \sum_{i=1}^4 B_i^5(\bar{u}_k)\mathbf{V}_i \right) = 0$$

which indicates that

$$\sum_{i=1}^4 \left(\sum_{k=1}^{N-1} B_l^5(\bar{u}_k)B_i^5(\bar{u}_k) \right) \mathbf{V}_i = \sum_{k=1}^{N-1} B_l^5(\bar{u}_k)\mathbf{Q}_k \tag{8}$$

Equation (8) is one linear equation in the unknowns $\mathbf{V}_1, \dots, \mathbf{V}_4$. For $l = 1, \dots, 4$ yields the system of 4 equations in 4 unknowns. For solving purpose, Eq. (8) can be written in matrix form

$$\left(B^T B \right) \mathbf{V} = \mathbf{Q} \tag{9}$$

where B is the $(N-1) \times 4$ matrix of scalars

$$B = \begin{bmatrix} B_1^5(\bar{u}_1) & \dots & B_4^5(\bar{u}_1) \\ \vdots & \ddots & \vdots \\ B_1^5(\bar{u}_{N-1}) & \dots & B_4^5(\bar{u}_{N-1}) \end{bmatrix}$$

\mathbf{Q} is the vector of $N-1$ points.

$$\mathbf{Q} = \begin{bmatrix} B_1^5(\bar{u}_1)\mathbf{Q}_1 + \dots + B_1^5(\bar{u}_{N-1})\mathbf{Q}_{N-1} \\ \vdots \\ B_4^5(\bar{u}_1)\mathbf{Q}_1 + \dots + B_4^5(\bar{u}_{N-1})\mathbf{Q}_{N-1} \end{bmatrix}$$

and $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_4 \end{bmatrix}$

It is noted that Eq. (9) is one coefficient matrix, with two set of unknowns (x, y coordinates). In order to solve Eq. (5) for the control points $\mathbf{V}_i (i = 1, \dots, 4)$, the

host parameter values \bar{u}_k are determined based on chord length parameterisation. The resulting quintic Bézier curve is presented below.

Automated approximation of airfoil shape by quintic Bézier curve

This part of work discusses the approximation results of air-foil by quintic Bézier curve. The resulting control points are given by $V_0(1.9,1)$, $V_1(-0.36,0.87)$, $V_2(1.81,5.98)$, $V_3(4.35, -0.55)$, $V_4(4.32, 2.27)$, $V_5(6.1,1)$ as shown in Fig. 8.

The maximum error of 0.17 mm occurs at the flatter region as shown in Fig. 9, which does not meet the tolerance of 0.1 mm. In term of curve shape, there is an undesirable fit at the highly curve region and irregular transition at the flatter region as can be seen in Fig. 9. Compare to the manual approximation as discussed in Section 5, the result generated by automated approximation is not promising. This is due to the fact that the automated method does not able to capture the small change of local geometry. In this sense, the automated method performs in global fitting, where only overall error is taken into account regardless of local shape change. Even though the use of automated method is practically more convenient in curve approximation, the resulting approximation is not necessarily better than the manual method.

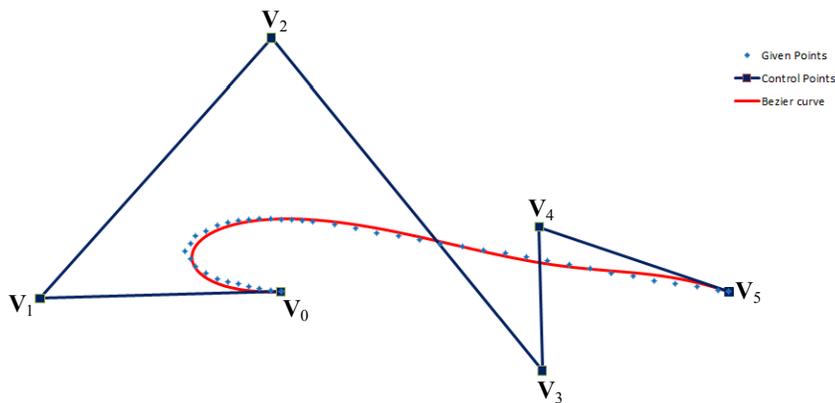


Fig. 8. Resulting Automated Approximation of Airfoil Shape.

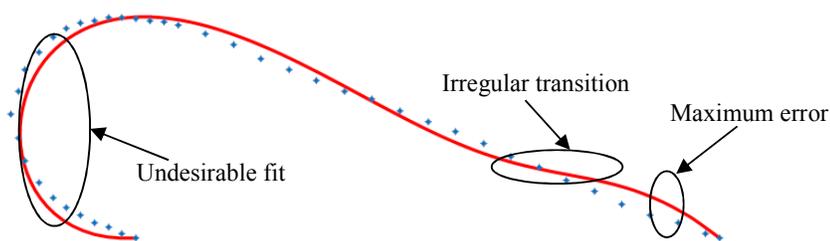


Fig. 9. Enlargement View of Automated Approximation of Airfoil Shape.

5. Conclusion

The approximation of airfoil shape in cubic Bézier curve and quintic Bézier curve in manual method are discussed. Both the cubic and quintic Bézier curves approximations are able to satisfy the tolerance of 0.1 mm. However, cubic Bézier curve does not have sufficient degree of freedom to capture the shape of highly curve region. quintic Bézier curve provides second derivative control, whereby the shape of highly curve region can be captured.

It should be emphasised that the consideration of airfoil shape approximation is not limited to satisfy the tolerance. Furthermore, the shape capturing capability needs to be taken into consideration. In conclusion, manual method approximation of quintic Bézier curve provides better shape approximation than that of cubic Bézier curve.

The automated approximation of quintic Bézier curve has also been explored. The result is not promising if compare to that of manual method. It is due to the automated method only limited to approximate the curve by minimising overall error regardless of the nature of curve shape. Therefore, it can be concluded that the automated approximation is not necessarily better than the manual approximation.

So far, the automated method is only based on minimising overall error. This is insufficient to produce desirable curve shape. To improve the automated method, further iterative mechanism need to be included with the consideration of local shape characteristic.

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