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ON MAGNETOROTATORY DOUBLE-DIFFUSIVE CONVECTION COUPLED WITH CROSS-DIFFUSIONS

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Abstract

The present paper investigates the effect of a uniform vertical rotation and a uniform vertical magnetic field on the physical problem of double-diffusive convection coupled with cross-diffusions. The governing equations of the problem are linearized by the construction of a proper transformation and some general qualitative results concerning the stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude are derived. The results for the double-diffusive convection problems with or without the individual consideration of Dufour and Soret effects follow as consequence.

Keywords: Double-diffusive convection, Dufour-Soret effects, Rayleigh numbers, Prandtl numbers, Taylor number, Chandrasekhar number.

1. Introduction

Thermosolutal convection or more generally double-diffusive convection, like its classical counterpart, namely, single-diffusive convection, has carved a niche for itself in the domain of hydrodynamic stability on account of its interesting complexities as a double-diffusive phenomenon as well as its direct relevance in the fields of Oceanography, Astrophysics, Geophysics, Limnology and Chemical engineering etc. can be seen from the review articles by Turner [1] and Brandt and Fernando [2]. An interesting early experimental study is that of Caldwell [3]. The problem is more complex than that of a single-diffusive fluid because the gradient in the relative concentration of two components can contribute to a density gradient just as effectively as can a temperature gradient. Further, the presence of two diffusive modes allows either stationary or overstable flow states

Nomenclatures

a	Dimensionless wave number
D_T	Dufour number
D_T	Dufour number
d	Depth of layer, m
g	Gravitational acceleration, m/s^2
$\vec{h}$	Magnetic field, G
P	Pressure, Pa
p	Growth rate, 1/s
Q	Chandrasekhar number
$\vec{q}$	Velocity, m/s
R_T	Thermal Rayleigh number
R_S	Solute Rayleigh number
S_T	Soret number
T	Taylor number
t	Time, s

Greek Symbols

α	Coefficient of thermal expansion, 1/K
α'	Coefficient of solute expansion, 1/K
β	Uniform temperature gradient, K/m
β'	Uniform concentration gradient, K/m
η	Electrical resistivity, m^2/s
θ	Perturbation in temperature, K
κ	Thermal diffusivity, m^2/s
κ'	Mass diffusivity, m^2/s
ν	Kinematic viscosity, m^2/s
ρ	Density, kg/m^3
σ	Prandtl number
σ_1	Magnetic Prandtl number
τ	Lewis number
ϕ	Perturbation in concentration, Kg

at the onset of convection depending on the magnitude of the fluid parameters, the boundary conditions and the competition between thermal expansion and the thermal diffusion. More complicated double-diffusive phenomenon appears if the destabilizing thermal/concentration gradient is opposed by the effect of a magnetic field or rotation. In the domain of linear stability theory the double-diffusive convection problems can be described by a set of linear ordinary differential equations with constant coefficient and homogeneous boundary conditions. The task of finding the explicit analytical solutions of these equations (especially when boundaries are rigid) and thereby characterizing the critical conditions at the threshold of instability are not entirely trivial since prohibitive amount of numerical work is required to affirm oscillatory or non- oscillatory motions as the Eigen value equation involves all the parameters of the problem implicitly. The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects [4, 5]. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and identification is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. Mohan [6, 7] mollified the nastily behaving governing equations of Dufour- driven thermosolutal convection and Soret-driven thermosolutal convection problems of the Veronis [8] type by the construction of a linear transformation and derived the desired results concerning the linear growth rate and the behavior of oscillatory motions on the lines suggested by Banerjee et al. [9, 10]. The analysis of double-diffusive convection becomes complicated in case when the diffusivity of one property is much greater than the other. Further, when two transport processes take place simultaneously, they interfere with each other and produce cross diffusion effects. The Soret and Dufour coefficients describe the flux of mass caused by temperature gradient and the flux of heat caused by concentration gradient respectively. The coupling of the fluxes of the stratifying agents is a prevalent feature in multicomponent fluid systems. In general, the stability of such systems is also affected by the cross-diffusion terms. Generally, it is assumed that the effect of cross diffusions on the stability criteria is negligible. However, there are liquid mixtures for which cross diffusions are of the same order of magnitude as the diffusivities. There are only few studies available on the effect of cross diffusion on double diffusion convection largely because of the complexity in determining these coefficients. Hurle and Jakeman [11] have studied the effect of Soret coefficient on the double-diffusive convection. They have reported that the magnitude and sign of the Soret coefficient were changed by varying the composition of the mixture. McDougall [12] has made an in depth study of double-diffusive convection where in both Soret and Dufour effects are important.

Motivated by these considerations the present paper investigates the instability of magnetorotatory double-diffusive convection problem coupled with cross-diffusions and derives some general qualitative results concerning the stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude. The results for the double-diffusive convection problems with or without the individual consideration of Dufour and Soret effects follow as a consequence.

2. Mathematical Formulation of the Problem

Following the usual steps of linear stability theory, the non-dimensional linearized perturbation equations governing the magnetorotatory double-diffusive convection problem coupled with cross-diffusions with a uniform rotation and a uniform magnetic field both acting in the vertical direction opposite to the force field of gravity are given by [8, 13].

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T\, a^2\theta - R_s a^2\phi - QD(D^2 - a^2)h_z + TD\zeta \qquad (1)$$

$$\left(D^2 - a^2 - p\right)\theta + D_T(D^2 - a^2)\,\phi = -w \qquad (2)$$

$$\left(D^2 - a^2 - \frac{p}{\tau}\right)\phi + S_T (D^2 - a^2)\,\theta = -\frac{w}{\tau} \qquad (3)$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw \qquad (4)$$

$$\left(D^2 - a^2 - \frac{p}{\sigma}\right)\zeta = -Dw - QD\xi \qquad (5)$$

and

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)\xi = -D\zeta \qquad (6)$$

with

$$w = 0 = D^2 w = \theta = \phi = D\zeta = h_z = D\xi \text{ at } z = 0 \text{ and } z = 1 \qquad (7)$$

$$w = 0 = Dw = \theta = \phi = \zeta = h_z = D\xi \text{ at } z = 0 \text{ and } z = 1 \qquad (8)$$

or any other combination of Eqs. (7) and (8).

In Eqs. (1)-(8), z is real independent variable such that $0 \le z \le 1$, $D=d/dz$ is differentiation with respect to z, $a^2>0$ is a constant, $\sigma>0$ is a constant, $\sigma_1 > 0$ is a constant, $\tau > 0$ is a constant, R_T and R_S are positive constants for the Veronis' configuration, $T > 0$ is a constant, $Q > 0$ is a constant, $p = p_r + ip_i$ is complex constant in general and the dependent variables $w, \theta, \phi, h_z, \zeta, \xi$ are complex valued functions of real variable z. The meanings of symbols from the physical point of view are as follows; z is the vertical coordinate, d/dz is differentiation along the vertical direction, a^2 is square of horizontal wave number, $\sigma = \upsilon/\kappa$ is the thermal Prandtl number, $\sigma_1 = \upsilon/\eta$ is the magnetic Prandtl number, $\tau = \eta_1/\kappa$ is the Lewis number, $R_T = g\alpha\beta_1 d^4/\kappa\upsilon$ is the thermal Rayleigh number, $R_S = g\alpha\beta_2 d^4/\kappa\upsilon$ is the concentration Rayleigh number, $T = 4\Omega^2 d^4/\nu^2$ is the Taylor number, $Q = \mu_e H^2 d^4/4\pi\rho\nu\eta$ is the Chandrasekhar number, $D_T = \beta_2 D_f/\beta_1\kappa$ is the Dufour number, $S_T = \beta_1 S_f/\beta_2\eta_1$ is the Soret number, ϕ is the concentration, θ is the temperature, p is the complex growth rate, w is the vertical velocity, h_z is the vertical magnetic field, ζ is the vertical vorticity and ξ is the vertical current density respectively.

3. Linear Transformation and Mathematical Analysis

The nature of the system (1)-(5) is clearly qualitatively different from those of double-diffusive convection problems ($D_T = 0 = S_T$) as now we have coupling between all the three eigen- functions $w, \theta,$ and ϕ in these equations. Consequently, they behave nastily and obstruct any attempt for the elegant extension of the earlier results for the double-diffusive convection problems to the present generalized set up. The nasty behaviour of these equations is mollified by the linear transformations given by:

$\tilde{w} = (S_T + B)\, w$

$\tilde{\theta} = E\theta + F\phi$

$\tilde{\phi} = S_T\theta + B\phi$

$\tilde{\zeta} = (\, S_T + B)\, \zeta$

$\tilde{\xi} = (\, S_T + B)\, \xi$

where

$$B = -\frac{1}{\tau}A,\;\; E = \frac{S_T + B}{D_T + A}A,\;\; F = \frac{S_T + B}{D_T + A}D_T$$

and A is a positive root of the equation

$$A^2 + (\tau - 1)A - \tau\, S_T D_T = 0\,.$$

The system of equations (1)-(6) together with boundary conditions (7)-(8), upon using the transformations as defined above takes the following forms:

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T'\, a^2\theta - R_S' a^2\phi - QD\left(D^2 - a^2\right)h_z + TD\zeta \qquad (9)$$

$$\left\{k_1\left(D^2 - a^2\right) - p\right\}\theta = -w \qquad (10)$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw \qquad (12)$$

$$\left(D^2 - a^2 - \frac{p}{\sigma}\right)\zeta = -Dw - QD\xi \qquad (13)$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)\xi = -D\zeta \qquad (14)$$

with

$$w = 0 = D^2 w = \theta = \phi = D\zeta = h_z = D\xi \text{ at } z = 0 \text{ and } z = 1 \qquad (15)$$

$$w = 0 = Dw = \theta = \phi = \zeta = h_z = D\xi \text{ at } z = 0 \text{ and } z = 1 \qquad (16)$$

where $k_1 = 1 + \frac{\tau\, D_T S_T}{A}$, $k_2 = 1 - \frac{S_T D_T}{A}$ are positive constants and

$R_T' = \frac{(D_T + A)(R_T B + R_S S_T)}{BA - S_T D_T}$, $R_S' = \frac{(S_T + B)(R_S A + R_T D_T)}{BA - S_T D_T}$ are respectively the modified thermal Rayleigh number and the modified concentration Rayleigh number. The sign tilde has been omitted for simplicity.

The system of Eqs. (9)-(14) together with either of the boundary conditions (15)-(16) constitutes a characteristics value problem for p for given values of the other parameters namely, $R_T', R_S', a^2, \sigma, \tau, T$ and a given state of the system is stable, neutral or unstable according as p_r the real part of p, is negative, zero or positive. Further, if $p_r = 0 \Rightarrow p_i = 0$ for all wave numbers a^2, then the principal of exchange of stabilities (PES) is valid otherwise we have overstability at least when instability sets in as certain modes.

We now prove the following theorems:

Theorem 1: If $(p, w, \theta, \phi, h_z, \xi, \zeta)$, $p = p_r + ip_i$, $p_i \neq 0$ is a non-trivial solution of (9)-(14) together with either of the boundary conditions (15)-(16) with $R'_T > 0$, $R'_S > 0$ and $J \leq 1$ then $p_r < 0$, where $J = \frac{4R'_T}{27\pi^4(1+\lambda)\,k_1}$ and $\lambda = \min\left\{\frac{\tau\, k_2}{\sigma}, \frac{1}{\sigma_1}, 1\right\}$.

Proof: Multiplying (9) by w^* (the complex conjugate of w) and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2 - a^2)\,(D^2 - a^2 - \frac{p}{\sigma})\,w\,dz - R'_T a^2 \int_0^1 \theta\, w^*\,dz + R'_S a^2 \int_0^1 \phi\, w^*\,dz + \\ + Q\int_0^1 w^* D\left(D^2 - a^2\right)h_z - T\int_0^1 w^*\,D\zeta = 0 \tag{17}$$

Now using equations (10)-(14) in the above equation, integrating the resulting equation a suitable number of times, using the relevant boundary conditions, we obtain

$$\int_0^1\left(\left|D^2 w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + \frac{p}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz \\ - R'_T a^2\int_0^1 [k_1\left(|D\theta|^2 + a^2|\theta|^2\right) + p^*|\theta|^2\ ]dz - R'_S a^2\,\tau\int_0^1 [k_2(\ |D\phi|^2 + a^2|\phi|^2) + \frac{p^*}{\tau}|\phi|^2\ ]dz \\ Q\int_0^1\left(\left|D^2 h_z\right|^2 + 2a^2|Dh_z|^2 + a^4|h_z|^2\right)dz + \frac{Q\,p^*\sigma_1}{\sigma}\int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + \\ + T\int_0^1\left(|D\zeta|^2 + a^2 + \frac{p^*}{\sigma}|\zeta|^2\right)dz + TQ\int_0^1\left(|D\xi|^2 + a^2|\xi|^2 + \frac{p^*\sigma_1}{\sigma}|\xi|^2\right)dz = 0 \tag{18}$$

Equating the real and imaginary parts of (18) equal to zero and using $p_i \neq 0$, we get

$$\int_0^1\left(\left|D^2 w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + \frac{p_r}{\sigma}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz \\ - R'_T a^2\int_0^1 [k_1\left(|D\theta|^2 + a^2|\theta|^2\right) + p_r|\theta|^2\ ]dz + R'_S a^2\,\tau\int_0^1 [k_2(\ |D\phi|^2 + a^2|\phi|^2) + \frac{p_r}{\tau}|\phi|^2\ ]dz \\ + Q\int_0^1\left(\left|D^2 h_z\right|^2 + 2a^2|Dh_z|^2 + a^4|h_z|^2\right)dz + \frac{Q\,p_r\sigma_1}{\sigma}\int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + \\ + T\int_0^1\left(|D\zeta|^2 + a^2 + \frac{p_r}{\sigma}|\zeta|^2\right)dz + TQ\int_0^1\left(|D\xi|^2 + a^2|\xi|^2 + \frac{p_r\sigma_1}{\sigma}|\xi|^2\right)dz = 0 \tag{19}$$

and

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz + R_T' a^2 \int_0^1 |\theta|^2 dz - R_S' a^2 \int_0^1 |\phi|^2 dz - \frac{T}{\sigma}\int_0^1 |\zeta|^2 - \frac{Q\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2 \right)dz + \frac{Q\sigma_1}{\sigma} T\int_0^1 |\xi|^2 dz = 0 \tag{20}$$

Multiplying Eq. (20) by p_r and adding the resulting equation to Eq. (19)

$$\int_0^1 \left(|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2 \right)dz - R_T' a^2 \int_0^1 \left[k_1 \left(|D\theta|^2 + a^2|\theta|^2 \right) \right]dz + R_S' a^2 \tau \int_0^1 \left[k_2 \left(|D\phi|^2 + a^2|\phi|^2 \right) \right]dz + \frac{2p_r}{\sigma}\int_0^1 |Dw|^2 + a^2|w|^2)dz + T\int_0^1 \left(|D\zeta|^2 + a^2|\zeta|^2 \right)dz + Q\int_0^1 \left(|D^2 h_z|^2 + 2a^2|Dh_z|^2 + a^4|h_z|^2 \right)dz + TQ\int_0^1 \left(|D\xi|^2 + a^2|\xi|^2 \right)dz + \frac{2p_r\sigma_1 TQ}{\sigma_1}\int_0^1 |\xi|^2 dz = 0 \tag{21}$$

Now, it is clear from Eq. (21) that

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz - \frac{Q\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2 \right) dz < R_S' a^2 \int_0^1 |\phi|^2 dz \tag{22}$$

or

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz - R_S' a^2 \int_0^1 |\phi|^2 < \frac{Q\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2 \right) dz \tag{23}$$

or

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2 + a^2|w|^2)dz < R_S' a^2 \int_0^1 |\phi|^2 + \frac{Q\sigma_1}{\sigma}\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2 \right) dz \tag{24}$$

Since w, θ, ϕ, h_z *and* ζ vanish at z = 0 and z = 1, therefore Rayleigh-Ritz inequality [13] yields

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz \tag{25}$$

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz \tag{26}$$

$$\int_0^1 |D\phi|^2 dz \geq \pi^2 \int_0^1 |\phi|^2 dz \tag{27}$$

$$\int_0^1 |Dh_z|^2 dz \geq \pi^2 \int_0^1 |h_z|^2 dz \tag{28}$$

$$\int_0^1 |D\zeta|^2 \, dz \geq \pi^2 \int_0^1 |\zeta|^2 \, dz \tag{29}$$

From inequalities (25) and (27) we have

$$\int_0^1 (|Dw|^2 + a^2|w|^2) dz \geq (\pi^2 + a^2) \int_o^1 |w|^2 dz \tag{30}$$

$$\int_0^1 (|D\phi|^2 + a^2|\phi|^2) dz \geq (\pi^2 + a^2) \int_o^1 |\phi|^2 dz \tag{31}$$

Further, utilizing Schwartz inequality, we have

$$\int_0^1 \left(|w^2|\right)^{\frac{1}{2}} \int_0^1 \left(|Dw|^2\right)^{\frac{1}{2}} \geq -\int_0^1 \left|w^* D^2 w\right| = \int_0^1 |Dw|^2 \geq \pi^2 \int_0^1 |w|^2$$

which on simplification yields

$$\int_0^1 \left(\left|D^2 w\right|^2 \right) \geq \pi^4 \int_0^1 |w|^2 \tag{32}$$

Inequality (30) together with inequality (32) yields

$$\int_0^1 \left(\left|D^2 w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2 \right) dz \geq (\pi^2 + a^2)^2 \int_0^1 |w|^2 dz \tag{33}$$

Multiplying (10) by its complex conjugate and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 \left[(k_1(D^2 - a^2) - p)\theta (k_1(D^2 - a^2) - p^*)\theta^* \right] dz = \int_0^1 ww^* dz$$

Integrating the above equation by parts an appropriate number of times and using either of the boundary conditions, we get

$$\int_0^1 \left|k^2{}_1(D^2 - a^2)\theta\right|^2 + 2p_r k \int_0^1 \left(|(D\theta)|^2 + a^2|\theta|^2 \right) dz + |p|^2 \int_0^1 |\theta|^2 dz = \int_0^1 |w|^2 dz. \tag{34}$$

Since $p_r \geq 0$, therefore from (34), we have

$$k_1^2 \int_0^1 \left|(D^2 - a^2)\theta\right|^2 dz \leq \int_0^1 |w|^2 dz \tag{35}$$

Also emulating the derivation of inequalities (30) and (33), we derive the following inequality

$$\int_0^1 \left|\left(D^2 - a^2 \right)\theta\right|^2 dz = \int_0^1 \left|D^2\theta\right|^2 + 2a^2\left|D\theta\right|^2 + a^4\left|\theta\right|^2 dz \geq \left(\pi^2 + a^2\right)^2 \int_0^1 \left|\theta\right|^2 dz \qquad (36)$$

Combining inequalities (35) and (36), we get

$$\int_0^1 |w|^2\, dz \geq \left(\pi^2 + a^2\right)^2 k_1^{\,2} \int_0^1 |\theta|^2 dz \qquad (37)$$

Also, we know $\int_0^1 |w|^2 = \int_0^1 \left(\left|w^2\right|\right)^{\frac{1}{2}} \int_0^1 \left(\left|w\right|^2\right)^{\frac{1}{2}}$ which upon using inequalities (35) and (36) yields

$$\int_0^1 |w|^2 dz \geq k_1^{\,2}\left(\pi^2 + a^2\right)\left\{\int_0^1 \left|\left(D^2 - a^2\right)\theta\right|^2 dz\right\}^{\frac{1}{2}} \sqrt{\int_0^1 |\theta|^2 \, dz} \qquad (38)$$

$$\geq k_1^{\,2}\left(\pi^2 + a^2\right)\left|-\int_0^1 \theta * \left(D^2 - a^2\right)\theta \, dz\right| \qquad \text{(Using Schwartz inequality)}$$

$$\geq \left(\pi^2 + a^2\right) k_1^{\,2} \int_0^1 \left\{\left|D\theta\right|^2 + a^2\left|\theta\right|^2\right\} dz \qquad (39)$$

Further, using Rayleigh –Ritz inequality; namely

$$\int_0^1 \left(\left|D^2 h_z\right|^2\right) \geq \pi^2 \int_0^1 \left|Dh_z\right|^2 \qquad (40)$$

$$\int_0^1 \left(\left|Dh_z\right|^2\right) \geq \pi^2 \int_0^1 \left|h_z\right|^2 \qquad (41)$$

we can have

$$\int_0^1 \left(\left|D^2 h_z\right|^2 + 2a^2\left|Dh_z\right|^2 + a^4\left|h_z\right|^2\right) dz \geq \left(\pi^2 + a^2\right)\int_0^1 \left(\left|Dh_z\right|^2 + a^2\left|h_z\right|^2\right) dz \qquad (42)$$

Now using inequalities (23), (30), (31), (33), (39)-(42), Eq. (21) yields

$$\left(\pi^2 + a^2\right)^2 \int_0^1 |w|^2 dz + \frac{2p_r}{\sigma}\int_0^1 \left(\left|Dw^2\right| + a^2|w|^2\right) dz + R_S' a^2 \tau\, k_2 \left(\pi^2 + a^2\right)\int_0^1 |\phi|^2 dz$$
$$+ \left(\pi^2 + a^2\right) Q \int_0^1 \left(\left|Dh_z\right|^2 + a^2\left|h_z\right|^2\right) dz + TQ\int_0^1 \left(\left|D\xi\right|^2 + a^2|\xi|^2\right) dz + T\left(\pi^2 + a^2\right)\int_0^1 |\zeta|^2 dz \qquad (43)$$
$$+ \frac{2p_r\sigma_1}{\sigma} TQ \int_0^1 |\xi|^2 dz < k_1 R_T' a^2 \int_0^1 \left(\left|D\theta\right|^2 + a^2|\theta|^2\right) dz$$

Using inequality (39) and the fact that $p_r \geq 0$, inequality (43) yields

$$\begin{aligned}&\left(\pi^2+a^2\right)^2\int_0^1|w|^2dz+R_S'a^2\tau\,k_2\left(\pi^2+a^2\right)\int_0^1|\phi|^2dz\\&+\left(\pi^2+a^2\right)Q\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz+T\left(\pi^2+a^2\right)\int_0^1|\zeta|^2dz<\frac{R_T'}{\left(\pi^2+a^2\right)k_1}a^2\int_0^1|w|^2dz\end{aligned} \tag{44}$$

Now, Eq. (20) upon using Rayleigh-Ritz inequality yields either of the three inequalities;

$$\frac{\left(\pi^2+a^2\right)}{\sigma}\int_0^1|w|^2d<R_S'a^2\int_0^1|\phi|^2dz+\frac{T}{\sigma}\int_0^1|\zeta|^2dz+\frac{Q\sigma_1}{\sigma}\int_0^1\left(\;|Dh_z|^2+a^2|h_z|^2\;\right)dz \tag{45}$$

$$Q\int_0^1\left(\;|Dh_z|^2+a^2|h_z|^2\;\right)dz>\frac{\left(\pi^2+a^2\;\right)}{\sigma_1}\int_0^1|w|^2)dz-\frac{R_S'a^2\sigma}{\sigma_1}\int_0^1|\phi|^2dz-\frac{T}{\sigma}\int_0^1|\zeta|^2dz \tag{46}$$

and

$$T\int_0^1|\zeta|^2dz>\left(\pi^2+a^2\right)\int_0^1|w|^2dz-R_S'a^2\sigma\int_0^1|\phi|^2dz-Q\sigma_1\int_0^1\left(\;|Dh_z|^2+a^2|h_z|^2\;\right)dz \tag{47}$$

Using inequalities (45)-(47), in inequality (44), respectively, we get the following inequalities namely;

$$\begin{aligned}&\left(\pi^2+a^2\right)^2\left(1+\frac{\tau k_2}{\sigma}\right)\int_0^1|w|^2dz+T\left(1-\frac{\tau k_2}{\sigma}\right)\left(\pi^2+a^2\right)\int_0^1|\zeta|^2dz\\&+Q\left(1-\frac{\tau k_2\sigma_1}{\sigma}\right)\left(\pi^2+a^2\right)\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz<\frac{R_T'a^2}{k_1\left(\pi^2+a^2\right)}\int_0^1|w|^2dz\end{aligned} \tag{48}$$

$$\begin{aligned}&\left(\pi^2+a^2\right)^2\left(1+\frac{1}{\sigma_1}\right)\int_0^1|w|^2dz+T\left(1-\frac{1}{\sigma_1}\right)\left(\pi^2+a^2\right)\int_0^1|\zeta|^2dz\\&+\left(\tau\,k_2-\frac{\sigma}{\sigma_1}\right)\left(\pi^2+a^2\right)R_S'a^2\int_0^1\left(|\phi|^2\;\right)dz<\frac{R_T'a^2}{k_1\left(\pi^2+a^2\right)}\int_0^1|w|^2dz\end{aligned} \tag{49}$$

and

$$\begin{aligned}&2\left(\pi^2+a^2\right)^2\int_0^1|w|^2dz+R_S'a^2\left(\tau\,k_2-\sigma\right)\left(\pi^2+a^2\right)\int_0^1|\phi|^2dz\\&+\left(1-\sigma_1\right)\left(\pi^2+a^2\right)Q\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz<\frac{R_T'a^2}{k_1\left(\pi^2+a^2\right)}\int_0^1|w|^2dz\end{aligned} \tag{50}$$

Now, let $\lambda=\min\left\{\frac{\tau\,k_2}{\sigma},\frac{1}{\sigma_1},1\right\}$, then depending upon the value of λ, exactly one of the inequalities (48), (49) and (50) will imply that

$$\left\{\frac{k_1\left(\pi^2+a^2\right)^3}{a^2}\left(1+\lambda\right)-R_T'\right\}\int_0^1|w|^2dz<0 \tag{51}$$

which implies that

$$\frac{(1+\lambda)\left(\pi^2+a^2\right)^3 k_1}{R_T' a^2} < 1 \tag{52}$$

Since the minimum value of $\left(\pi^2+a^2\right)^3/a^2$ with respect to a^2 is $27\pi^4/4$, it follows from inequality (52) that $\frac{27\pi^4 k_1(1+\lambda)}{4R_T'} < 1$, i.e., J>1, where $J = \frac{4R_T'}{27\pi^4 k_1(1+\lambda)}$.

This is a contradiction to the given hypothesis of the theorem. Hence, we must have $p_r < 0$. This completes the proof of the theorem.

The above theorem in the terminology of hydrodynamic stability theory implies that for the problem under consideration arbitrary oscillatory perturbations of growing amplitude are not allowed if $J \le 1$.

Corollary 1. For rotatory hydromagnetic double-diffusive convection, if if R_T>0, R_S>0, p_i≠0 and J_1≤1, then p_r<0, where

$$J_1 = \frac{4R_T}{27\pi^4(1+\lambda_1)} \text{ and } \lambda_1 = \min\left\{\frac{\tau}{\sigma}, \frac{1}{\sigma_1}\right\}.$$

Corollary 2. For rotatory hydromagnetic Soret-driven double-diffusive convection (D_T=0) if R_T>0, R_S>0, p_i≠0 and J_2≤1, then p_r<0, where

$$J_2 = \frac{4\left(R_T - \frac{\tau R_T S_T}{(1-\tau)}\right)}{27\pi^4(1+\lambda_1)} \text{ and } \lambda_1 = \min\left\{\frac{\tau}{\sigma}, \frac{1}{\sigma_1}\right\}.$$

Corollary 3. For rotatory hydromagnetic Dufour-driven double-diffusive convection (S_T=0) if R_T>0, R_S>0, p_i≠0 and J_3≤1, then p_r<0, where

$$J_3 = \frac{4R_T\left\{1++\frac{D_T}{(1-\tau)}\right\}}{27\pi^4(1+\lambda_1)} \text{ and } \lambda_1 = \min\left\{\frac{\tau}{\sigma}, \frac{1}{\sigma_1}\right\}.$$

Remark: We note that if J>1, then oscillatory modes of growing amplitudes can exist. Further, keeping in view Theorem 1 and the fact that the growth rate p has been intentionally avoided in the proof of this theorem, one strongly feels that a bound for the growth rate of oscillatory motions of growing amplitude in terms of the parameters of the problem specifically involving (J-1) as factor must be derivable. The subsequent theorem justifies our intuition.

Theorem 2. If (p, w, θ, ϕ, ζ, ξ, h_z), $p = p_r + ip_i$, $p_i \neq 0$ is a non-trivial solution of Eqs. (9)-(14) together with either of the boundary conditions (15)-(16) with $R_T' > 0$ and $R_S' > 0$, then

$|p| < \dfrac{R'_T\sqrt{J^2-1}}{4\pi^2(1+\lambda)}$, where

$J = \dfrac{4R'_T}{27\pi^4(1+\lambda)\,k_1}$ and $\lambda = \min\left\{\dfrac{\tau\, k_2}{\sigma}, \dfrac{1}{\sigma_1}1\right\}$.

Proof. Proceeding exactly as in Theorem1, utilizing the fact that $p_r \geq 0$, we have from Eq. (21)

$$\int_0^1\left(\left|D^2w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + R'_S a^2\,\tau\int_0^1\left[k_2\left(|D\phi|^2 + a^2|\phi|^2\right)\right]dz + Q\left(\pi^2+a^2\right)$$
$$\int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + T\int_0^1\left(|D\zeta|^2 + a^2|\zeta|^2\right) < R'_T a^2\int_0^1\left[k_1\left(|D\theta|^2 + a^2|\theta|^2\right)\right]dz \tag{53}$$

From (34) it follows that

$$\int_0^1\left|k^2{}_1\left(D^2-a^2\right)\theta\right|^2 + |p|^2\int_0^1|\theta|^2\,dz \leq \int_0^1|w|^2\,dz \tag{54}$$

Using inequality (36) in inequality (54), we get

$$\int_0^1|w|^2\,dz \geq \left(\pi^2+a^2\right)^2 k_1^{\,2}\left\{1 + \frac{|p|^2}{k_1^{\,2}\left(\pi^2+a^2\right)^2}\right\}\int_0^1|\theta|^2\,dz \tag{55}$$

Now,

$$\int_0^1\left\{|D\theta|^2 + a^2|\theta|^2\right\}dz = \left|-\int_0^1\theta^*\left(D^2-a^2\right)\theta\right|dz \leq \int_0^1|\theta|\left|\left(D^2-a^2\right)\theta\right|\,dz$$

$$\leq\left\{\int_0^1\left|\left(D^2-a^2\right)\theta\right|^2dz\right\}^{\frac{1}{2}}\sqrt{\int_0^1|\theta|^2} \qquad \text{(Using Schwartz inequality)}$$

$$\leq\frac{1}{k_1^{\,2}\left(\pi^2+a^2\right)^2}\left\{1+\frac{|p|^2}{k_1^{\,2}\left(\pi^2+a^2\right)^2}\right\}^{\frac{-1}{2}}\left\{\int_0^1|w|^2dz\right\} \tag{56}$$

(using inequalities (54) and (55))

Making use of inequalities (30), (31), (33), (42) and (56) in (53), we have

$$\left(\pi^2+a^2\right)^2\left(1+\frac{\tau k_2}{\sigma}\right)\int_0^1|w|^2dz + T\left(\pi^2+a^2\right)\left(1-\frac{\tau\,k_2}{\sigma}\right)\int_0^1|\zeta|^2\,dz +$$
$$Q\left(\pi^2+a^2\right)\left(1-\frac{\tau k_2\sigma_1}{\sigma}\right)\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz \leq \frac{R'_T a^2}{k_1\left(\pi^2+a^2\right)\sqrt{1+\dfrac{|p|^2}{k_1^{\,2}\left(\pi^2+a^2\right)^2}}}\int_0^1|w|^2dz \tag{57}$$

$$\left(\pi^2+a^2\right)^2\left(1+\frac{1}{\sigma_1}\right)\int_0^1|w|^2dz+T\left(1-\frac{1}{\sigma_1}\right)\left(\pi^2+a^2\right)\int_0^1|\zeta|^2dz$$
$$+\left(\tau\,k_2-\frac{\sigma}{\sigma_1}\right)\left(\pi^2+a^2\right)R_S'a^2\int_0^1\left(|\phi|^2\right)dz<\frac{R_T'a^2}{k_1\left(\pi^2+a^2\right)\sqrt{1+\frac{|p|^2}{k_1^2\left(\pi^2+a^2\right)^2}}}\int_0^1|w|^2dz \tag{58}$$

and

$$2\left(\pi^2+a^2\right)\int_0^1|w|^2dz+R_S'a^2\left(\pi^2+a^2\right)\left(\tau\,k_2-\sigma\right)\int_0^1|\phi|^2dz\left(1-\sigma_1\right)\left(\pi^2+a^2\right)Q\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz$$
$$\leq\frac{R_T'a^2}{k_1\left(\pi^2+a^2\right)\sqrt{1+\frac{|p|^2}{k_1^2\left(\pi^2+a^2\right)^2}}}\int_0^1|w|^2dz \tag{59}$$

Now, let $\lambda=\min\left\{\frac{\tau\,k_2}{\sigma},\frac{1}{\sigma_1},1\right\}$, then depending on the value of λ, exactly one of the inequalities (57) -(59)will imply that

$$k_1(1+\lambda)\frac{\left(\pi^2+a^2\right)^3}{a^2}\int_0^1|w|^2<R_T'\left[1+\frac{|p|^2}{k_1^2\left(\pi^2+a^2\right)^2}\right]^{-\frac{1}{2}}\int_0^1|w|^2dz \tag{60}$$

Since, minimum value of $\left(\pi^2+a^2\right)^3/a^2$ with respect a^2 is $27\pi^4/4$, it therefore follows from inequality (60) that

$$|p|<k_1\left(\pi^2+a^2\right)\sqrt{J^2-1} \tag{61}$$

where J=$\frac{4R_T'}{27\pi^4(1+\lambda)k_1}$. Further, since $\left[1+\frac{|p|^2}{k_1^2\left(\pi^2+a^2\right)^2}\right]>1$, therefore from inequality (60), we can have

$$\frac{\left(\pi^2+a^2\right)^3}{a^2}k_1(1+\lambda)<R_T' \tag{62}$$

Since, minimum value of $\left(\pi^2+a^2\right)^2/a^2$ with respect to a^2 is $4\pi^2$, therefore inequality (62) yields

$$\left(\pi^2+a^2\right)<\frac{R_T'}{4\pi^2(1+\lambda)\,k_1} \tag{63}$$

Combining inequalities (62) and (63), we get $|p|<\frac{R_T'}{4\pi^2(1+\lambda)}\sqrt{J^2-1}$. This completes the proof of the theorem.

Theorem 2 from the point of view of hydrodynamic stability theory may be stated as: The complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) for the problem under consideration lies inside a semi- circle in the right-half of the $p_r p_i$ -plane whose centre is at the origin and whose radius is $\frac{R_T'\sqrt{J^2-1}}{4\pi^2(1+\lambda)}$.

Corollary 4. For rotatory hydromagnetic double-diffusive convection ($D_T = S_T = 0$), the complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) lies inside a semi- circle in the right-half of the $p_r p_i$ -plane whose centre is at the origin and whose radius is

$$\frac{R_T\sqrt{J_1^{\,2}-1}}{4\pi^2(1+\lambda_1)}$$

where $J_1 = \frac{4R_T}{27\pi^4(1+\lambda_1)}$ and $\lambda_1 = \min\left\{\frac{\tau}{\sigma}, 1\right\}$.

Corollary 5. For rotatory hydromagnetic Soret-driven double-diffusive convection ($D_T = 0$), the complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) lies inside a semi-circle in the right-half of the $p_r p_i$ -plane whose centre is at the origin and whose radius is

$$\frac{\left\{R_T - \frac{\tau R_T S_T}{(1-\tau)}\right\}}{4\pi^2(1+\lambda_1)}\sqrt{J_2^{\,2}-1}$$

where $J_2 = \frac{4\left\{R_T - \frac{\tau R_T S_T}{(1-\tau)}\right\}}{27\pi^4(1+\lambda_1)}$ and $\lambda_1 = \min\left\{\frac{\tau}{\sigma}, 1\right\}$.

Corollary 6. For rotatory hydromagnetic Dufour-driven double-diffusive convection ($S_T = 0$) the complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) lies inside a semi- circle in the right-half of the $p_r p_i$ -plane whose centre is at the origin and whose radius is

$$\frac{R_T\left(1+\frac{D_T}{(1-\tau)}\right)\sqrt{J_3^{\,2}-1}}{4\pi^2(1+\lambda_1)},$$ where

$$J_3 = \frac{4R_T\left\{1++\frac{D_T}{(1-\tau)}\right\}}{27\pi^4(1+\lambda_1)}$$ and $\lambda_1 = \min\left\{\frac{\tau}{\sigma}, 1\right\}$.

4. Results and Discussion

The problem of double-diffusive convection coupled with cross-diffusion under the simultaneous effect of rotation and magnetic field has been investigated. Double-diffusive fluid systems correspond to mixture wherein the concentration and thermal diffusive modes (Soret and Dufour effects) are absent. When these coupled effects are also taken into account, the stability of such systems is also affected by the cross-diffusion terms. Although, it is assumed that the effect of cross-diffusions on the stability criteria is negligible, however, there are liquids mixtures for which cross-diffusions are of same order of magnitude as the diffusivities. The analysis made herein brings out the following results:

- In the terminology of hydrodynamic stability theory, Theorem 1 implies that for the problem under consideration arbitrary oscillatory perturbations of growing amplitude are not allowed if $J = \frac{4R'_T}{27\pi^4 k_1(1+\lambda)} \le 1$. It is to be noted here that if $J > 1$, and then oscillatory modes of growing amplitude can exist. Keeping in view of the result of Theorem 1 one strongly feels that a bound for the growth rate of oscillatory motions of growing amplitude in terms of the parameters of the problem specifically involving (J-1) as factor must be derivable and Theorem 2 is in this direction.
- The complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) for the problem under consideration lies inside a semi- circle in the right half of the $p_r p_i$-plane whose centre is at origin and whose radius is $\frac{R'_T\left(\sqrt{J^2-1}\right)}{4\pi^2(1+\lambda)}$.

The results for the double-diffusive convection problem under the simultaneous effect of rotation and magnetic field with or without the individual consideration of Dufour and Soret effects follow as a consequence.

5. Conclusions

The main contribution of the present study is to investigate the influence of two effects, namely, the Dufour effect and the Soret effect on the magnetorotatory double-diffusive convection problem. The investigation of cross-diffusion effect is motivated by its interesting complexities as a double-diffusive phenomenon which has its importance in various fields such as high quality crystal production, oceanography, production of pure medication, solidification of molten alloys, exothermally heated lakes and magmas. Followings are the main conclusions:

- The oscillatory motions of growing amplitude are not allowed in magnetorotatory double-diffusive convection coupled with cross-diffusions if $J = \frac{4R'_T}{27\pi^4 k_1(1+\lambda)} \le 1$.
- The complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) perturbation of growing amplitude ($p_r \geq 0$) for the problem under consideration

lies inside a semi-circle in the right half of the $p_r p_i$-plane whose centre is at origin and whose radius is $\dfrac{R_T'\left(\sqrt{J^2-1}\right)}{4\pi^2(1+\lambda)}$.

- The present study is confined to horizontal layer geometry on account of complexity of the problem for arbitrary geometry. However, there exists a class of results in the domain of hydrodynamic and hydromagnetic stability theory that possess the sparks of their generalization to containers of arbitrary shape. This is an open problem and is hoped to be resolved in the near future.

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