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AN EFFICIENT BOUNDARY FITTED NON-HYDROSTATIC MODEL FOR FREE-SURFACE FLOWS

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Abstract

A boundary fitted non-hydrostatic finite volume model is presented to simulate two dimensional vertical free surface flows effectively deploying only 2-4 vertical layers. The algorithm is based on a projection method which results in a block tri-diagonal system of equation with pressure as the unknown. This system can be solved by a direct matrix solver without iteration. To purpose of minimise the computational cost, a new top-layer pressure treatment is proposed which enables the model to simulate relatively short wave motion with very small vertical layers accurately. The test of linear and nonlinear sinusoidal short wave propagation with significant vertical accelerations is applied correctly using a small number of layers.

Keywords: Implicit Finite Volume Method, Projection Method, Non-Hydrostatic Pressure, Free-Surface Flow, Top Layer Treatment.

1. Introduction

Long waves such as tides with periods in the order of hours can be successfully simulated assuming a hydrostatic pressure distribution in depth. This assumption is no more valid when short waves with periods in the order of few seconds are to be simulated and deploying a dynamic pressure distribution is necessary. Although depth-integrated models based on Mild-slope and Boussinesq Equations have been successfully applied to simulate short wave propagation they are

Nomenclatures

A	Wave amplitude
c	Wave celerity
g	Gravitational acceleration
h	Water depth
i	The number of cells in longitudinal direction
M	Total number of cells in longitudinal direction
N	Total number of cells in vertical direction
n	Number of time step
nk	Total number of layers
P	Total Pressure
P^*	Excess Pressure
T	Wave period
t	Time
u	Component of velocity in the x direction
u^*	Intermediate velocity in the x direction
u^{**}	Second intermediate velocity in the x direction
w	Component of velocity in the z direction
w^*	Intermediate velocity in the z direction
w^{**}	Second intermediate velocity in the z direction
x	Coordinate in horizontal direction
z	Coordinate in vertical direction
Δt	Numerical time step
Δx	Numerical space increment in x direction
Δz	Numerical space increment in z direction

Greek Symbols

ρ	Water density
ν_t	Eddy viscosity coefficient
ξ	Surface elevation
θ, ψ	Implicit weighting factors

unable to predict the variation of flow structure within depth. In order to obtain a better understanding of these variations, 2DV and 3D models should be applied.

Such non-hydrostatic models have been developed recently by Mahadevan et al. [1] and Li and Fleming [2]. A fully non-hydrostatic implicit algorithm was suggested by Namin et al. [3] and Yuan and Wu [4, 5]. The algorithm solves the Navier–Stokes equations and the free-surface boundary condition simultaneously and forms a block tri-diagonal system with the unknown horizontal velocity. Ahmadi et al. [6] developed an implicit finite volume two-dimensional in vertical plane (2DV) model on a sigma coordinate like mesh to simulate free surface flows. The algorithm, based on projection method, solved the complete 2DV NSE with pressure as unknown.

In non-hydrostatic models that use hydrostatic pressure assumption at the top layer with staggered grid mesh, a large number of vertical layers are necessary to simulate non-hydrostatic free-surface flows [1, 2 and 3]. To remove the top-layer hydrostatic assumption, Yuan and Wu [4] planned an integral technique in sigma coordinate framework. Their results demonstrate that by applying non-hydrostatic pressure distribution to the top layer, phase errors are noticeably reduced in the simulation of dispersive waves. Stelling and Zijlema [7] presented an approximation of vertical gradient of the non-hydrostatic pressure based on the Keller-box or Preissmann scheme. Their results show that this procedure allows a very small number of layers (in the order of 1–3) for the simulation of relatively short waves. Recognizing most models are based upon a staggered grid system, following Stelling and Zijlema [7], Yuan and Wu [5] from sigma coordinate mesh switched to Cartesian coordinate and proposed an integral method, different to the Keller-box scheme, to obtain a non-hydrostatic pressure condition at the free surface cell. Using a small number of vertical layers, they also show the model can accurately predict very steep waves. Subsequently Choi and Wu [8] changed the numerical algorithm to projection method in which, the size of resulting matrix, is a quarter smaller than pervious work.

In this paper, a boundary fitted finite volume model for solving the complete 2DV NSE in the free-surface flows is presented. In order to minimise the computational cost, a new top-layer pressure treatment is proposed. This method enables the model to simulate relatively short wave motion with very small vertical layers accurately. The algorithm is based on time splitting method which results in a block tri-diagonal system of equation with pressure as the unknown. This system of equations can be solved by a direct matrix solver without iteration. With this algorithm, the water elevation can be obtained along with the velocity and pressure fields as a part of the solution. Boundary fitted grid system is chosen as the computational mesh, which enables the model to simulate free-surface flows over irregular geometries. The model is validated by two tests using very small vertical layers, namely a linear progressive short wave and nonlinear progressive wave in intermediate depth. To validate the model, numerical results are compared with analytical solutions and experimental data.

2. Mathematical Formulation

The governing equations used to describe the two-dimensional vertical, incompressible flows are continuity and momentum equations. After division pressure P into two parts named, the 'hydrostatic pressure' and 'excess pressure' as $P = -\rho g z + \rho P^*$, the conservative form of the equations is expressed as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial P^*}{\partial x} = \frac{\partial}{\partial x}\left(\nu_t \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial u}{\partial z}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial P^*}{\partial z} = \frac{\partial}{\partial x}\left(\nu_t \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial w}{\partial z}\right) \tag{3}$$

where t is time; x, z are coordinates in horizontal and vertical directions respectively; u, w are components of velocity in the x and z directions respectively; ρ is the density of water; g is the gravitational acceleration; and ν_t is the eddy viscosity coefficient.

3. Boundary Conditions

The kinematic boundary condition at the impermeable bottom is $(u\frac{\partial h}{\partial x}+w=0)$ where h is the water depth. At impermeable bottom and wall boundaries zero normal velocity is applied. For viscous flows no-slip boundary condition is considered. In non viscous flows, tangential velocity gradient at wall boundaries is set to zero. Similarly, the kinematic boundary condition at the moving free surface is $(\frac{\partial \xi}{\partial t}+u\frac{\partial \xi}{\partial x}=w)$ where ξ is the surface elevation. Atmospheric pressure is assumed at free-surface elevation. Measured laboratory or theoretical velocity distribution in the vertical direction are adopted as inflow boundary conditions. Sommerfeld radiation boundary condition together with sponge layer technique [5] is implemented at outflow boundaries to eliminate wave reflection.

4. Numerical Methods

A finite volume approximation is used to discretise the governing equations and boundary conditions. A boundary fitted staggered grid mesh system with a set of M × N cells respectively in longitudinal and vertical direction is employed. The mesh is a special non-orthogonal curvilinear system, which fits the surface and bottom boundaries of the domain. The major assumptions for this system are the constant number of vertical grids along the whole domain and the curvature applying only in the longitudinal direction. Figure 1 illustrates the location of the main variables.

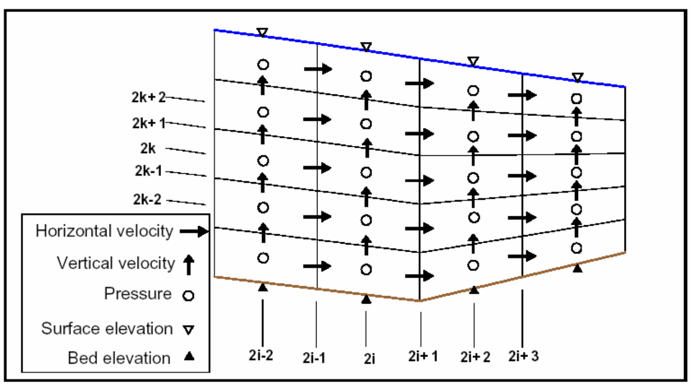


Fig. 1. Presentation of Staggered Grids and the Positions of the Variables.

A fractional step algorithm (projection method) is deployed to solve the governing equations in two major steps:

1. Step I: In the first step the pressure term in the momentum equations is excluded and the resultant advection–diffusion equations are solved [6]. This step is divided into two stages. In stage one velocities are advected using the known velocity field at the previous time step *n*, to obtain the new intermediate velocity field u^* and w^*. In the next stage the diffusion terms are solved to find second intermediate velocities u^{**} and w^{**}.

2. Step II: In the second step the continuity Eq. (1), together with the momentum equations without advection and diffusion terms are solved simultaneously.

The details of finite volume derivatives and numerical discretisation can be found in Ahmadi et al [6] and are not given here for simplicity.

The following equation can be written for the mass continuity for the column *2i*:

$$\frac{\xi_{2i}^{n+1}-\xi_{2i}^{n}}{\Delta t}+\frac{1}{\Delta x}\sum_{k=1}^{nk}\left(\theta\Delta z_{2i+1}u_{2i+1,2k}^{n+1}-\theta\Delta z_{2i-1}u_{2i-1,2k}^{n+1}+(1-\theta)\Delta z_{2i+1}u_{2i+1,2k}^{n}-(1-\theta)\Delta z_{2i-1}u_{2i-1,2k}^{n}\right)=0 \tag{4}$$

where nk is the number of layers and the implicit weighting factor θ, is taken as 0.5. Δz_{2i+1} refers to the projected length of the $(2i+1)th$ edge in the *z* direction. Vertical momentum equation with only pressure term in column *2i* from the centre of top layer, to the free surface is approximated by:

$$\frac{w_{2i.T}^{n+1}-w_{2i.T}^{**}}{\Delta t}+\psi\left(\frac{P_{2i.S}^{*n+1}-P_{2i.2nk}^{*n+1}}{\Delta z_{2i}/2}\right)+(1-\psi)\left(\frac{P_{2i.S}^{*n}-P_{2i.2nk}^{*n}}{\Delta z_{2i}/2}\right)=0 \tag{5}$$

where $P_{2i.S}^{*n+1}=g\xi_{2i}^{n+1}$ and $P_{2i.S}^{*n}=g\xi_{2i}^{n}$ are pressure at surface water level. ψ is taken as 0.5. $w_{2i.T}^{n+1}$ is vertical velocity at top-layer located in distance of $0.25\Delta z_{2i}$ from the surface. Yuan and Wu [4] suggested $w_{2i.T}^{n+1}=(w_{2i.2nk-1}^{n+1}+w_{2i.2nk+1}^{n+1})/2$. However their model yields good results by employing large number of vertical layers (in case of linear sinusoidal short wave 20 layers). It seems linear approximation $w_{2i.T}^{n+1}=0.25w_{2i.2nk-1}^{n+1}+0.75w_{2i.2nk+1}^{n+1}$ is a better selection with respect to location of $w_{2i.T}^{n+1}$. Yet $w_{2i.T}^{n+1}$ can be approximated with higher order accuracy. For instance, in the case of a third order approximation the following relationship can be obtained.

$$w_{2i.T}^{n+1}=\frac{77}{128}w_{2i.2nk+1}^{n+1}+\frac{77}{128}w_{2i.2nk-1}^{n+1}-\frac{33}{128}w_{2i.2nk-3}^{n+1}+\frac{7}{128}w_{2i.2nk-5}^{n+1} \tag{6}$$

Substituting $w_{2i.T}^{n+1}$ from Eq. (6); ξ_{2i}^{n+1} from equation (4); $u_{2i-1.2k}^{n+1}$ and $w_{2i.2k-1}^{n+1}$ from horizontal and vertical momentum equations (without advection and diffusion terms); and $w_{2i.2nk+1}^{n+1}$ from continuity equation, into Eq. (5), the velocities are eliminated and the pressure equation of the upper layer is obtained.

By writing this, an equation for the top layer is obtained together with the pressure equation of the lower layers a tri-diagonal block matrix system is formed which can be solved by a direct matrix solver. The details of this procedure and solution method are described in [6]. After the solution of unknown pressure, horizontal flow velocity *u*, vertical velocity *w* and free surface elevation are updated by back substitution.

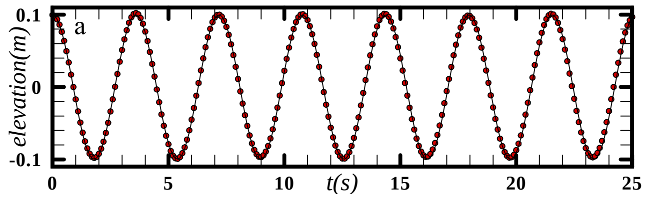
5. Model Validation

5.1. Linear standing short wave in deep water

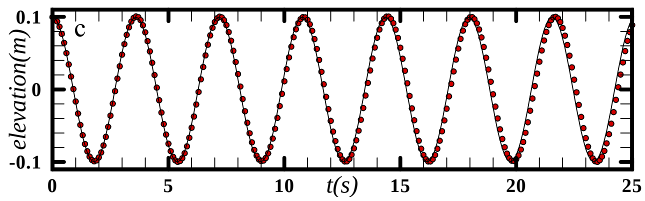
A linear standing short wave in deep water assuming an inviscid flow is the first test representing the effects of hydrodynamic pressure distribution on model results. The uninodal wave oscillates in a 10 m long and 10m deep closed reservoir with amplitude of *A*=0.1m. The analytical solution for linear wave period and celerity is *T*=3.588s and *c*=5.575 m/s respectively. The linear wave theory is valid in the case. Details of analytical solutions of the linear standing wave can be found in [9]. The numerical parameters used in this test case are: $\Delta x = 0.5m$, $\Delta t = 0.05s$. As mentioned before in "top layer treatment" three approximations can be used for vertical velocity in top layer: approximation proposed by Yuan and Wu [4], linear approximation and 3rd-order approximation. Different results obtained by applying these three approximation at the top layer are presented in Fig. 2. Weak approximations cause noticeable phase error especially with using small number of vertical layer. On the contrary, 3^{rd}-order approximation yields a more accurate simulation of wave celerity with using only four vertical layers. CPU time of simulation decreases by a factor of twelve with four, instead of 20 vertical layers.

5.2. Nonlinear wave propagation in a constant water depth

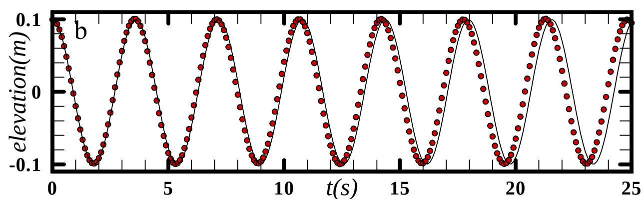
To validation of dynamic pressure and nonlinear behavior predicted by the model, another test including nonlinear progressive short wave in intermediate water depth assuming an inviscid flow is performed. The wave propagates from left to right along a 35.54 m long, 40 cm deep wave flume, with the initial water elevation being set to zero. A sinusoidal velocity distribution, with wave amplitude of A=4.2 cm, and T=2.5s is enforced at the left boundary condition. $\Delta x = 0.1m$, is grid distance in longitudinal direction. $\Delta t = 0.01s$ and N=2 has chosen as time interval and number of vertical layers within this test. Numerical results of free-surface elevation along the flume has compared against experimental data by [10]. In Figure 3 the numerical prediction shows close results with respect to experimental data. Close to the left boundary at x=1 meter, the wave profile is quasi-sinusoidal. At increasing distances an asymmetry develops that forms a small secondary crest in the through of the primary wave at x=7.0 meters. The progression along the channel discloses a slow return to quasi-sinusoidal profile at x=14.0 meters. This structure of length equal to almost three times the wave length reappears beyond this point, [10].



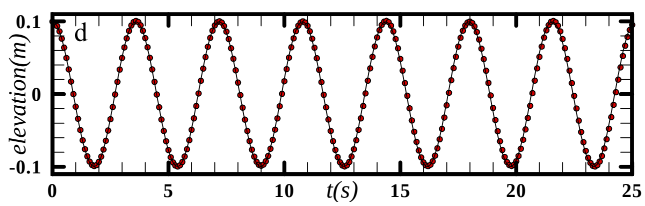
(a) Top layer approximation proposed by Yuan and Wu [4] with 20 vertical layers.



(b) Top layer approximation proposed by Yuan and Wu [4] with 4 vertical layers.



(c) Linear top layer approximation with 4 vertical layers.



(d) 3rd-order top layer approximation with 4 layers.

Fig. 2. Comparisons of free-surface elevation for a linear standing wave test (x=0.25 m), between analytical solutions (solid lines) and numerical results (dots) from different models

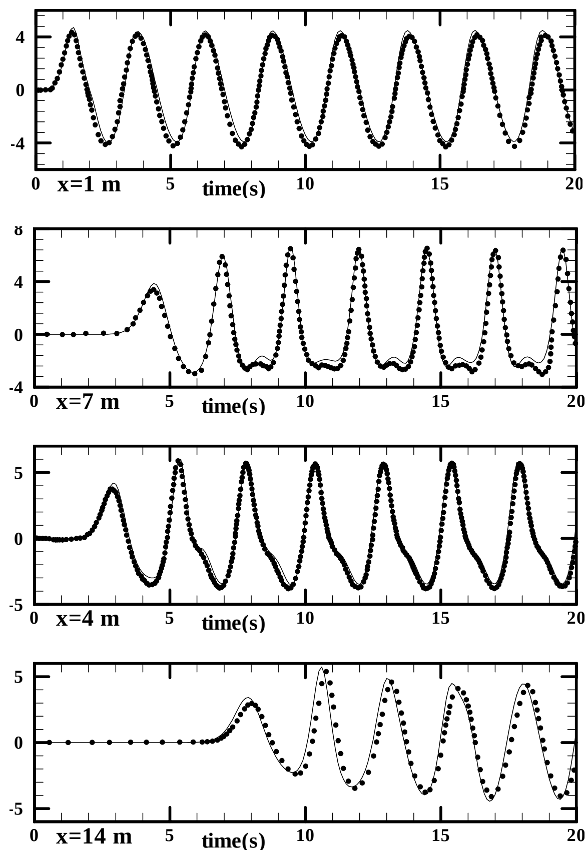


Figure 3: Comparisons of the free-surface elevation for nonlinear sinusoidal wave propagation along the wave channel between numerical results (solid line) and experimental data (circles). The amplitudes are expressed in cm.

6. Conclusions

In present work an efficient fully dynamic model based on Finite volume method is developed to simulate free surface flow in a two dimensional vertical plane. The boundary fitted grid layout is chosen as the computational mesh, which enables the model to simulate free-surface flows over irregular geometries. The new treatment of non-hydrostatic pressure at the top layer makes the model to simulate complicated free-surface flow problems with a very small number of vertical layers accurately and free of any hydrostatic pressure assumption. The numerical solution is time splitting method which solves the equations in two major steps. The resultant system of equations can be solved directly without iteration in an efficient way. The matrix system has a quarter smaller of models which solve equations in one step simultaneously [8]. In addition, different numerical schemes for each step can be used within this algorithm.

To validate the model, two tests, including a complicated free surface, were performed with only two or four vertical layers. In modeling standing linear short wave, it was shown by deploying new top layer treatment, the number of vertical layers can be reduced noticeably (20 layers to 4 layers). This treatment causes the computational cost decreases severely. To evaluate the capability of model for simulating nonlinear wave with significant vertical acceleration, the second test applied and demonstrated good results with two vertical layers.

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