Journal of Engineering Science and Technology
Vol. 16, No. 3 (2021) 2018 - 2027

DISCRETE ORLICZ-MORREY SPACES AND THEIR INCLUSION PROPERTIES

SITI FATIMAH[1,*], AL AZHARY MASTA[2],
SOfiHARA AL HAZMY[3], CECE KUSTIAWAN[4] AND INDRA RUKMANA[5]

[1,2,4]Department of Mathematics Education, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi 229, Bandung 40154, Indonesia
[3]Department of Mathematics, Universitas Pertahanan Republik Indonesia, Kawasan IPSC Sentul, Sukahati, Bogor, Jawa Barat 16810
[5]Senior High School, SMAIT Rhaudatul Jannah Cilegon, Residence Grand Cilegon, Cilegon, Banten 42426
*Corresponding Author: sitifatimah@upi.edu

Abstract

There are two types of Orlicz space already known, i.e namely discrete Orlicz space and 'continuous' Orlicz space. Those spaces have been intensively studied by many researchers in the last few decades. The aim of this paper is to define discrete Orlicz-Morrey spaces as a generalization of discrete Orlicz space, especially the structure of discrete Orlicz-Morrey spaces by using the Luxemburg norm and computing the characteristic sequence norm as the keys for obtaining the inclusion properties of discrete Orlicz-Morrey spaces. The main results of this research are sufficient and necessary conditions for the inclusion properties of these spaces. Based on our results, two discrete Orlicz-Morrey spaces are comparable concerning Young's function for integers set.

Keywords: Discrete Orlicz spaces, Discrete Orlicz-Morrey spaces, Inclusion property, Luxemburg norm.

1. Introduction

There are extension of Orlicz spaces and Morrey spaces i.e Orlicz-Morrey spaces. Many mathematician have made important observations about Orlicz spaces and Morrey spaces, for example [1-8]. Besides the 'continuous' Orlicz spaces and 'continuous' Morrey spaces, several authors have made studies about discrete Orlicz spaces (see references [6, 9, 10]) and discrete Morrey spaces (see references [11, 12]).

Recently, Gunawan et al. [12] showed the inclusion properties on generalized discrete Morrey spaces and their weak type as in the following theorem.

Theorem 1.1. [12]. We let $1 \leq p_1 \leq p_2 < \infty, \phi_1 \in \mathcal{G}_1$ *and* $\phi_2 \in \mathcal{G}_2$. Then, statements (1) through (5) are equivalent:

$$\phi_2 \lesssim \phi_1 \ (on\ 2\omega + 1). \tag{1}$$

$$\ell^{p_2}_{\phi_2} \subseteq \ell^{p_1}_{\phi_1}. \tag{2}$$

$$\| \cdot \|_{\ell^{p_1}_{\phi_1}} \lesssim \| \cdot \|_{\ell^{p_2}_{\phi_2}} \left(on\ \ell^{p_2}_{\phi_2} \right). \tag{3}$$

$$\omega\ell^{p_2}_{\phi_2} \subseteq \omega\ell^{p_1}_{\phi_1}. \tag{5}$$

$$\| \cdot \|_{\ell^{p_1}_{\phi_1}} \lesssim \| \cdot \|_{\ell^{p_2}_{\phi_2}} \left(on\ \omega\ell^{p_2}_{\phi_2} \right). \tag{6}$$

The related result in discrete Orlicz spaces can be found in reference [9]. Based on these results, we are motivated to study the discrete Orlicz-Morrey space. Especially, we examine the inclusion properties of these spaces. For related results on the continuous version of Orlicz-Morrey spaces, we can see reference [13-15]. The novelty of this paper is to get the sufficient and necessary condition for inclusion properties of discrete Orlicz-Morrey spaces. From our results, we would like to find out what factors influence in the inclusion property of these spaces.

First, we rewrite the definition of Young's function. A function $\Phi: [0, \infty] \rightarrow [0, \infty]$ is said to be Young's function if it satisfies the following conditions:

a) $\Phi(tx + (1 - t)y) \leq t\Phi(x) + (1 - t)\Phi(y)$ for $0 \leq t \leq 1$.

b) Φ is continuous function

c) $\Phi(0) = 0$

d) $\lim_{t \to \infty} \Phi(t) = \infty$.

Let Φ, Ψ be Young's function, we write $\Phi \prec \Psi$ if $\Phi(t) \leq \Psi(Ct)$ for some $C > 0$ and all $t > 0$.

By supposing $m \in \mathbb{Z}, N \in \omega \coloneqq \mathbb{N} \cup \{0\}$, we notate $S_{m,N} \coloneqq \{m - N, \ldots, m, \ldots, m + N\}$. Then, $|S_{m,n}| = 2N + 1$ for its cardinality of $S_{m,N}$. Let G_ϕ denotes the set of all functions $\phi: 2\omega + 1 \rightarrow (0, \infty)$ which fulfills ϕ as nondecreasing and $\frac{\phi(2N+1)}{2N+1}$ as nonincreasing.

For $\phi_1, \phi_2: 2\omega + 1 \to (0, \infty)$, we write $\phi_1 \preccurlyeq \phi_2$ if a constant $C > 0$ is exist such that $\phi_1(2M + 1) \leq C\phi_2(2M + 1)$ for all $M \in \omega$. If $\phi_1 \preccurlyeq \phi_2$ and $\phi_2 \preccurlyeq \phi_1$, then we write $\phi_1 \approx \phi_2$.

Now, we gave Φ as a Young's function and $\phi \in G_\phi$. We state the discrete Orlicz-Morrey spaces $\ell_{\phi,\Phi}(\mathbb{Z})$ as a set of sequences $x = (x_k)_{k=1}^{\infty}$. By taking values in $\mathbb{Z}$ such that for every m element of $\mathbb{Z}$ and N elemet of ω, the following quantity is finite.

$$\|x\|_{\phi,\Phi,m,N} := \inf\left\{b > 0: \frac{\phi(2N+1)}{|S_{m,N}|} \sum_{k \in S_{m,N}} \Phi\left(\frac{|x_k|}{b}\right) \leq 1\right\} \tag{6}$$

In the main section, we show the discrete Orlicz-Morrey spaces $\ell_{\phi,\Phi}(\mathbb{Z})$ is a Banach space equipped the norm of $\|x\|_{\ell_{\phi,\Phi}} := \sup_{m \in \mathbb{Z}, N \in \omega} \|x\|_{(\phi,\Phi,m,N)}$. . For $\phi(2M + 1) = 2M + 1$, space $\ell_{\phi,\Phi}(\mathbb{Z})$ is the discrete Orlicz space $\ell_{\phi,\Phi}(\mathbb{Z})$. Aside from that, for $\Phi(t) = t^p$, the space $\ell_{\phi,\Phi}(\mathbb{Z})$ reduces to the generalized discrete Morrey space $\ell_\phi^p(\mathbb{Z})$.

The next of part of this paper is to organize mathematical derivations and their proofs. We present some lemmas which are useful for obtaining our results in section 2. In section 3, the main results are presented (i.e. we state some of the results regarding the inclusion property of Orlicz-Morrey discrete spaces). In this paper, the letter A denotes a constant whose value that can be different for each row. For constants with fixed values, we use subscripts such as A_1 and A_2.

2. Research Methodology

To obtain our results, we use some lemmas as in the following.

Lemma 2.1. [13, 14, 16], we gave Φ as a Young function and $\Phi^{-1}(a) := \inf\{r \geq 0 : \Phi(r) > a\}$ for every $s \geq 0$. Then the followings hold:

(1) $\Phi^{-1}(0) = 0$

(2) $\Phi^{-1}(a_1) \leq \Phi^{-1}(a_2)$ *for* $a_1 \leq a_2$.

(3) $\Phi\left(\Phi^{-1}(a)\right) \leq a \leq \Phi^{-1}\left(\Phi(a)\right)$ *for* $0 \leq a < \infty$.

(4) If for some constants $C_1, C_2 > 0$, we have $\Phi_2^{-1}(s) \leq C_1 \Phi_1^{-1}(C_2 s)$ and then

$$\Phi_1\left(\frac{t}{C_1}\right) \leq C_2 \Phi_2(t)$$

for $t = \Phi_2^{-1}(s)$. We use Lemma 2.1 for proving one of our results.

Lemma 2.2. we gave Φ as a Young function and $\phi \in G_\phi$. For $m_0 \in \mathbb{Z}$ *and* $N_0 \in \omega$, let ξ^{m_0,N_0} be the characteristics sequence given by

$$\xi^{m_0,N_0} := \begin{cases} 1, & \text{if } k \in S_{m_0,N_0} \\ 0, & \text{otherwise} \end{cases}$$

Then, we have $\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} = \dfrac{1}{\phi^{-1}\left(\frac{|S_{m,N}|}{\phi(2N_0+1)|S_{m_0,N_0}|}\right)}$.

In the proof, we let $A_{\Phi,m,n} := \left\{b > 0: \Phi\left(\frac{1}{b}\right) \leq \frac{|S_{m,N}|}{\phi(2N+1)|S_{m_0,N_0}|}\right\} 3\ R$, and $B_{\Phi,m,n} := \left\{r > 0: \Phi(r) > \frac{|S_{m,N}|}{\phi(2N+1)|S_{m_0,N_0}|}\right\}$. We observe that,

$$\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} = \inf\left\{b > 0: \frac{\Phi(2N+1)}{|S_{m,N}|} \sum_{k \in S_{m,N}} \Phi\left(\frac{|\xi^{m_0,N_0}|}{b}\right) \leq 1\right\}$$

$$= \inf\left\{b > 0: \frac{\Phi(2N+1)|S_{m_0,N_0}|}{|S_{m,N}|} \Phi\left(\frac{1}{b}\right) \leq 1\right\}$$

$$= \inf\left\{b > 0: \Phi\left(\frac{1}{b}\right) \leq \frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right\}$$

$$= \inf A_{\Phi,m,N}.$$

Meanwhile, $\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right) = \inf\left\{r \geq 0: \Phi(r) > \frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right\} = \inf B_{\Phi,m,N}$. By choosing $b = \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)}$. By Lemma 2.1 (3), we have $\Phi\left(\frac{1}{b}\right) = \Phi\left(\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)\right) \leq \frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}$. This shows that $\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} \leq \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)}$.

Now, if $\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} < \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)}$, we get $\frac{1}{\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N}} > \Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)$.

By using the definition of $\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)$, there exists a $r_1 \in B_{\Phi,m,N}$ such that $\frac{1}{\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N}} > r_1 \geq \Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)$.

Since $r_1 \in B_{\Phi,m,N}$, we obtain $\frac{1}{r_1} \notin A_{\Phi,m,N}$, and it shows that $\frac{1}{r_1} \leq \|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N}$. This contradicts the fact that $\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} > r_1$. Hence, we must have $\|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} = \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)}$, as desired.

Proposition 2.3. We gave Φ as a Young function and $\phi \in G_{\phi}$. For $m_0 \in \mathbb{Z}$ and $N_0 \in \omega$, we let ξ^{m_0,N_0} be the characteristics sequence given by

$\xi^{m_0,N_0} := \begin{cases} 1, & \text{if } k \in S_{m_0,N_0} \\ 0, & \text{otherwise} \end{cases}$. Then, we have $\|\xi^{m_0,N_0}\|_{\ell_{\phi,\Phi}} = \frac{1}{\Phi^{-1}\left(\frac{1}{\Phi(2N_0+1)}\right)}$.

In the proof, for any $m_0 \in \mathbb{Z}$ and $N_0 \in \omega$, by Lemma 2.2, we have

$$\|\xi^{m_0,N_0}\|_{\ell_{\phi,\Phi}} = \sup_{m\in\mathbb{Z},N\in\omega} \|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N} = \sup_{m\in\mathbb{Z},N\in\omega} \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)}$$

$$\geq \frac{1}{\Phi^{-1}\left(\frac{1}{\Phi(2N_0+1)}\right)}.$$

Now, we prove $\|\xi^{m_0,N_0}\|_{\ell_{\phi,\Phi}} \leq \frac{1}{\Phi^{-1}\left(\frac{1}{\Phi(2N_0+1)}\right)}$.

The case I: for $N \geq N_0$, we have

$$\frac{\phi(2N+1)}{2N+1} \leq \frac{\phi(2N_0+1)}{2N_0+1} = \frac{\phi(2N_0+1)}{|S_{m_0,N_0}|}.$$

So we get,

$$\frac{1}{\phi(2N_0+1)} \leq \frac{2N+1}{|S_{m_0,N_0}|\phi(2N+1)} = \frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}.$$

By Lemma 2.1 (2), we obtain

$$\phi^{-1}\left(\frac{1}{\phi(2N_0+1)}\right) \leq \Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)$$

if and only if

$$\frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)} \leq \frac{1}{\phi^{-1}\left(\frac{1}{\phi(2N_0+1)}\right)}.$$

Case II: For $N < N_0$, we have $\phi(2N+1) \leq \phi(2N_0+1)$. It showed that

$$\frac{1}{\phi(2N_0+1)} \leq \frac{1}{\phi(2N+1)} \leq \frac{2N+1}{(2N_0+1)\phi(2N_0+1)} = \frac{|S_{m,N}|}{\Phi(2N_0+1)|S_{m_0,N_0}|}.$$

By Lemma 2.1 (2), we obtain $\phi^{-1}\left(\frac{1}{\phi(2N_0+1)}\right) \leq \Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)$ if the value $\frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)} \leq \frac{1}{\phi^{-1}\left(\frac{1}{\phi(2N_0+1)}\right)}$.

Since case I and II are true for arbitrary $m \in \mathbb{Z}, N \in \omega$, we have

$$\|\xi^{m_0,N_0}\|_{\ell_{\phi,\Phi}} = \sup_{m\in\mathbb{Z},N\in\omega} \|\xi^{m_0,N_0}\|_{\phi,\Phi,m,N}$$

$$= \sup_{m\in\mathbb{Z},N\in\omega} \frac{1}{\Phi^{-1}\left(\frac{|S_{m,N}|}{\Phi(2N+1)|S_{m_0,N_0}|}\right)} \leq \frac{1}{\Phi^{-1}\left(\frac{1}{\Phi(2N_0+1)}\right)},$$

3. Results and Discussion

First, we show $\|\cdot\|_{\ell_{\phi,\Phi}}$ that defines a norm on $\ell_{\phi,\Phi}(\mathbb{Z})$. For getting a result, we present some lemmas in the following.

Lemma 3.1. If $x \in \ell_{\phi,\Phi}(\mathbb{Z})$, then

$$\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{\|x\|_{\phi,\Phi,m,N}}\right) \leq 1$$

For any $m \in \mathbb{Z}$ and $N \in \omega$. Futhermore, $\|x\|_{\phi,\Phi,m,N} \leq 1$ if and only if ∑_($k \in S_{m,N}\Phi(|x_k|) \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$.

In the proof, we take an arbitrary $x \in \ell_{\phi,\Phi}(\mathbb{Z})$ and $\epsilon > 0$, then there exists $b_\epsilon > 0$ such that $\|x\|_{\phi,\Phi,m,N} \leq b_\epsilon \leq \|x\|_{\phi,\Phi,m,N} + \epsilon$ and $\frac{\Phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{b_\epsilon}\right) \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$. Since Φ is increasing, we have

$$\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{\|x\|_{\phi,\Phi,m,N}}\right) \leq \frac{\Phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{b_\epsilon}\right) \leq 1.$$

Since $\epsilon > 0$ is arbitrary, we conclude $\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{\|x\|_{\phi,\Phi,m,N}}\right) \leq 1$. Next, if $\|x\|_{\phi,\Phi,m,N} \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$, then

$$\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi(|x_k|) \leq \frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{\|x\|_{\phi,\Phi,m,N}}\right) \leq 1.$$

Now, we assume that $\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi(|x_k|) \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$. Clearly, we have $\|x\|_{\phi,\Phi,m,N} \leq 1$.

Lemma 3.2. Given Φ is a young function, $\phi \in G_\phi$, $m \in \mathbb{Z}$ and $N \in \omega$. Then the statements (1) and (2) are equivalent:

(1) $\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{a}\right) \leq 1$ for every $a > 0$.

(2) $\|x\|_{\phi,\Phi,m,N} = 0$.

In the proof, we assume that (1) holds. By definition of $\|x\|_{\phi,\Phi,m,N}$, we have $0 \leq \|x\|_{\phi,\Phi,m,N} \leq \epsilon$ for every $a > 0$. This shows that $\|x\|_{\phi,\Phi,m,N} = 0$. Suppose on the contrary, that there exist $a_0 > 0$ such that

$$\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{a_0}\right) > 1.$$

By Lemma 3.1 we have $\|x\|_{\phi,\Phi,m,N} \geq a_0 > 0$. As consequence, we conclude $\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_k|}{a}\right) \leq 1$, in which this is for every $a > 0$.

Lemma 3.3. We gave Φ as a Young function, $\phi \in G_\phi, m \in \mathbb{Z}$ and $N \in \omega$. Then statements (1) and (2) are equivalent:

(1) $\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi(\alpha|x_k|) = 0$ for every $\alpha > 0$.

(2) $\|x\|_{\phi,\Phi,m,N} = 0$.

In the proof, we suppose that (1) holds. As before, we can obtain $\|x\|_{\phi,\Phi,m,N} = 0$. Now, suppose that (2) holds. Take an arbitrary $0 < \epsilon \leq 1$. Since Φ is convex functions, we have

$$\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi(\alpha|x_k|) \leq \epsilon\left(\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi(\alpha|x_k|)\right) \leq \epsilon.$$

Since $0 < \epsilon \leq 1$ is arbitrary, we can conclude that

$$\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi(\alpha|x_k|) = 0.$$

We will use Lemma 3.3 to prove the following Proposition 3.4.

Proposition 3.4. We gave Φ as a Young function and $\phi \in G_\phi$, the function $\|\cdot\|_{\ell_{\phi,\Phi}}$ is a norm on $\ell_{\phi,\Phi}(\mathbb{Z})$. Moreover, $\ell_{\phi,\Phi}(\mathbb{Z})$ is a Banach space.

In the proof, it is clear $\|x\|_{\ell_{\phi,\Phi}} \geq 0$ and $\|ax\|_{\ell_{\phi,\Phi}} = |a|\|x\|_{\ell_{\phi,\Phi}}$ for every $x \in \ell_{\phi,\Phi}(\mathbb{Z})$ and $a \in \mathbb{Z}$. Now, we will show $\|x\|_{\ell_{\phi,\Phi}} = 0$ iff $x = 0$. If $x = 0$, then we get $\|x\|_{\ell_{\phi,\Phi}} = 0$. We let $\|x\|_{\ell_{\phi,\Phi}} = 0$, then $\|x\|_{\phi,\Phi,m,N} = 0$ is for every $m \in \mathbb{Z}$ and $N \in \omega$. By Lemma 3.3., we have $\frac{\phi(2N+1)}{|S_{m,n}|}\sum_{k\in S_{m,N}} \Phi(\alpha|x_k|) = 0$. In fact, $\phi(2N+1)$, $|S_{m,N}|$, and Φ has positive values, we have $x_k = 0$ for every $k \in S_{m,N}$. Since $m \in \mathbb{Z}$ and $N \in \omega$ are arbitrary, we have $x = 0$.

Next, we will show $\|x_1 + x_2\|_{\ell_{\phi,\Phi}} \leq \|x_1\|_{\ell_{\phi,\Phi}} + \|x_2\|_{\ell_{\phi,\Phi}}$ for every $x_1, x_2 \in \ell_{\phi,\Phi}(\mathbb{Z})$. We let $x_1 = (x_{1,k})_{k=1}^{\infty}$ and $x_2 = (x_{2,k})_{k=1}^{\infty}$ be elements of $\ell_{\phi,\Phi}$. For any $m \in \mathbb{Z}$ and $N \in \omega$, we get

$$\sum_{k\in S_{m,N}} \Phi\left(\frac{|x_{1,k} + x_{2,k}|}{\|x_1\|_{\phi,\Phi,m,N} + \|x_2\|_{\phi,\Phi,m,N}}\right)$$
$$\leq \sum_{k\in S_{m,N}} \Phi\left(\sum_{i=1}^{2} \frac{\|x_i\|_{\phi,\Phi,m,N}}{\|x_1\|_{\phi,\Phi,m,N} + \|x_2\|_{\phi,\Phi,m,N}} \cdot \frac{|x_{i,k}|}{\|x_i\|_{\phi,\Phi,m,N}}\right)$$

$$\leq \sum_{i=1}^{2}\left(\frac{\|x_i\|_{\phi,\Phi,m,N}}{\|x_1\|_{\phi,\Phi,m,N}+\|x_2\|_{\phi,\Phi,m,N}}\sum_{k\in S_{m,N}}\Phi\left(\frac{|x_{i,k}|}{\|x_i\|_{\phi,\Phi,m,N}}\right)\right) \leq \frac{|S_{m,N}|}{\phi(2N+1)}$$

Or, $\frac{\phi(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi\left(\frac{|(x_{1,k})(x_{2,k})|}{\|x_1\|_{\phi,\Phi,m,N}+\|x_2\|_{\phi,\Phi,m,N}}\right) \leq 1$. By definition of $\|x_1+x_2\|_{\phi,\Phi,m,N}$, we have $\|x_1+x_2\|_{\phi,\Phi,m,N} \leq \|x_1\|_{\phi,\Phi,m,N} + \|x_2\|_{\phi,\Phi,m,N}$. By taking supremum over $m \in \mathbb{Z}$ and $N \in \omega$, we conclude that $\|x_1+x_2\|_{\ell_{\phi,\Phi}} \leq \|x_1\|_{\ell_{\phi,\Phi}} + \|x_2\|_{\ell_{\phi,\Phi}}$ for every $x_1, x_2 \in \ell_{\phi,\Phi}(\mathbb{Z})$.

Now we come to inclusion of (strong) discrete Orlicz-Morrey spaces in the following.

Theorem 3.5. Let Φ_1, Φ_2 be Young's functions and $\phi_1, \phi_2 \in G_\phi$ such that $\phi_1 \preccurlyeq \phi_2$. Then statements (1) through (3) are equivalent:

(1) $\phi_1 \prec \phi_2$.

(2) $\ell_{\phi_2,\Phi_2}(\mathbb{Z}) \subseteq \ell_{\phi_1,\Phi_1}(\mathbb{Z})$.

(3) $\|x\|_{\ell_{\phi_1,\Phi_1}} \leq C\|x\|_{\ell_{\phi_2,\Phi_2}}$, for some $C > 0$ and every $x \in \ell_{\phi_2,\Phi_2}(\mathbb{Z})$.

In the proof, first, we will prove that (1) implies (2). Let $x \in \ell_{\phi_2,\Phi_2}(\mathbb{Z})$. Since $\Phi_1 \prec \Phi_2$ and $\phi_1 \preccurlyeq \phi_2$, there exist $C_1, C_2 > 0$ such that $\Phi_1(t) \leq \Phi_2(C_1 t)$ for every $t > 0$ and $\phi_1(2N+1) \leq C_2\phi_2(2N+1)$ for every $N \in \omega$. For every $m \in \mathbb{Z}$ and $N \in \omega$, we consider two cases in the following

Case I: For $0 < C_2 \leq 1$, we have

$$\begin{aligned}\frac{\phi_1(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_1\left(\frac{|x_k|}{C_1\|x\|_{\phi_2,\Phi_2,m,N}}\right) &\leq \frac{\phi_1(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_2\left(\frac{C_1|x_k|}{C_1\|x\|_{\phi_2,\Phi_2,m,N}}\right)\\ &\leq \frac{C_2\phi_2(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_2\left(\frac{|x_k|}{\|x\|_{\phi_2,\Phi_2,m,N}}\right)\\ &\leq \frac{\phi_2(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_2\left(\frac{|x_k|}{\|x\|_{\phi_2,\Phi_2,m,N}}\right) \leq 1.\end{aligned}$$

Case II: For $C_2 > 1$, we have

$$\begin{aligned}\frac{\phi_1(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_1\left(\frac{|x_k|}{C_1C_2\|x\|_{\phi_2,\Phi_2,m,N}}\right) &\leq \frac{\phi_1(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_2\left(\frac{C_1|x_k|}{C_1C_2\|x\|_{\phi_2,\Phi_2,m,N}}\right)\\ &\leq \frac{\phi_1(2N+1)}{C_2|S_{m,N}|}\sum_{k\in S_{m,N}}\Phi_2\left(\frac{|x_k|}{\|x\|_{\phi_2,\Phi_2,m,N}}\right)\end{aligned}$$

$$\leq \frac{\phi_1(2N+1)}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Phi_2\left(\frac{|x_k|}{\|x\|_{\phi_2,\Phi_2,m,N}}\right) \leq 1.$$

By definition of $\|x\|_{\phi_1,\Phi_1,m,N}$, we have $\|x\|_{\phi_1,\Phi_1,m,N} \leq \max\{1, C_2\}\,\|x\|_{\phi_2,\Phi_2,m,N}$. By taking supremum over $m \in \mathbb{Z}$ and $N \in \omega$, we conclude that $\|x\|_{\ell_{\phi_1,\Phi_1}} \leq \max\{1, C_2\}\,\|x\|_{\ell_{\phi_2,\Phi_2}}$.

This proves that $\ell_{\phi_2,\Phi_2}(\mathbb{Z}) \subseteq \ell_{\phi_1,\Phi_1}(\mathbb{Z})$. Next, since $\ell_{\phi_2,\Phi_2}(\mathbb{Z})$ and $\ell_{\phi_1,\Phi_1}(\mathbb{Z})$ are Banach space, by using Lemma 3.3 in reference [17] we have (2) and (3) are equivalent. We assume that above equation holds. Let $m_0 \in \mathbb{Z}$ and $N_0 \in \omega$. By Proposition 2.3, we have

$$\frac{1}{\Phi_1^{-1}\left(\frac{1}{\phi_1(2N_0+1)}\right)} = \|\epsilon^{m_0,N_0}\|_{\ell_{\phi_1,\Phi_1}} \leq C\|\epsilon^{m_0,N_0}\|_{\ell_{\phi_2,\Phi_2}} = \frac{C}{\Phi_2^{-1}\left(\frac{1}{\phi_2(2N_0+1)}\right)},$$

Hence, we can get $\Phi_2^{-1}\left(\frac{1}{\phi_2(2N_0+1)}\right) \leq C\Phi_1^{-1}\left(\frac{1}{\phi_1(2N_0+1)}\right)$. By Lemma 2.1 (4), we have

$$\Phi_1\left(\frac{t_0}{C}\right) \leq \Phi_2(t_0),$$

where $t_0 = \Phi_2^{-1}\left(\frac{1}{\phi_2(2N_0+1)}\right)$. Since $N_0 \in \omega > 0$ is arbitrary, we conclude that $\Phi_1(t) \leq \Phi_2(Ct)$ for every $t > 0$, as desired.

4. Conclusion

We have shown the inclusion properties of discrete Orlicz-Morrey spaces, by using the norm of the characteristic sequences in Z. as our final conclusion, we can state thet the condition Φ_1≺Φ_2 is sufficient and necessary conditions of inclusion properties on these spaces. There are some versions of continuous Orlicz-Morrey spaces already known. In the next research, we may obtain the necessary and sufficient conditions of inclusion on discrete Olicz-Morrey spaces and develop to patch current weak type.

References

1. Deringoz, F.; Guliyev, V.S.; and Samko, S. (2014). Boundedness of the maximal and singular operators on generalized Orlicz-Morrey spaces. *Operator Theory: Advances and Applications*, 242, 139-158.
2. Gala, S.; Sawano, Y.; and Tanaka, H. (2015). A remark on two generalized Orlicz-Morrey spaces. *Journal of Approximation Theory*, 198, 1-9.
3. Guliyev, V.S.; Hasanov, S.G.; Sawano, Y.; and Noi, T. (2016). Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind. *Acta Applicandae Mathematicae*, 145(1), 133-174.

4. Gunawan, H.; Hakim, D.I.; Limanta, K.M.; and Masta, A.A. (2017). Inclusion properties of generalized Morrey spaces. *Mathematische Nachrichten*, 290(2-3), 332-340.
5. Maligranda, L.; and Mastyło, M. (2000). Inclusion mappings between Orlicz sequence spaces. *Journal of Functional Analysis*, 176(2), 264-279.
6. Nowak, M. (1992). Linear functionals on Orlicz sequence spaces without local convexity. *International Journal of Mathematics and Mathematical Sciences*, 15(2), 241-254.
7. Sawano, Y.; Sugano, S.; and Tanaka, H. (2012). Orlicz-Morrey spaces and fractional operators. *Potential Analysis*, 36(4), 517-556.
8. Osançlıol, A. (2014). Inclusions between weighted Orlicz spaces. *Journal of Inequalities and Applications*, 2014(1), 1-8.
9. Ofie, M.; and Herawati, E. (2018). Inclusion Theorem Between the spaces generated by Musielak-ψ function. *Journal of Physics Conference Series*, 1116(2018) 022035, 1-7.
10. Prayoga, P.S.; Fatimah, S.; Masta, A.A. (2020). Sifat Inklusi Dan Perumuman Ketaksamaan Hölder Pada Ruang Barisan Orlicz. *Jurnal EurekaMatika*, 8(2), 91-109.
11. Berezhnoi, E.I. (2017). A discrete version of local Morrey spaces. *Izvestiya: Mathematics*, 81(1), 1-28.
12. Gunawan, H.; Kikianty, E.; and Schwanke, C. (2018). Discrete Morrey spaces and their inclusion properties. *Mathematische Nachrichten*, 291, 1283-1296
13. Masta, A.A.; Gunawan, H.; and Setya-Budhi, W. (2017). An inclusion property of Orlicz-Morrey spaces, *Journal of Physics Conference Series*, 893(1), 1-8.
14. Masta, A.A.; Gunawan, H.; and Setya-Budhi, W. (2017). On inclusion properties of two versions of Orlicz-Morrey spaces. *Mediterranean Journal of Mathematics*, 14(6), 1-12.
15. Deringoz, F.; Guliyev, V.S.; and Samko, S. (2017). Vanishing generalized Orlicz-Morrey spaces and fractional maximal operator. *Publicationes Mathematicae Debrecen*, 90(1-2), 125-147
16. Masta, A.A.; Gunawan, H.; and Setya-Budhi, W. (2016). Inclusion property of Orlicz and weak Orlicz spaces, *Journal of Mathematics and Fundamental Science*, 48, 193-203.
17. Krein, S.G.; Petunin, J.I.; and Semenov, E.M. (1982). Interpolation of linear operators, Translations of Mathematical Monographs, *American Mathematical Society Providence RI*, 54, 3-7.