Journal of Engineering Science and Technology
Vol. 12, No. 9 (2017) 2438 - 2452

PROBABILISTIC FINITE ELEMENT ANALYSIS OF A HEAVY DUTY RADIATOR UNDER INTERNAL PRESSURE LOADING

ROBIN ROY P.[1,*], V. HARIRAM[1], M. SUBRAMANIAN[2]

[1]Department of Mechanical Engineering,Hindustan Institute of Technology & Science, Hindustan University, Chennai - 603103, Tamil Nadu, India
[2]Department of Automobile Engineering, BS Abdur Rahman University, Chennai, India
*Corresponding Author: robin_cae@hotmail.com

Abstract

Engine cooling is vital in keeping the engine at most efficient temperature for the different vehicle speed and operating road conditions. Radiator is one of the key components in the heavy duty engine cooling system. Heavy duty radiator is subjected to various kinds of loading such as pressure, thermal, vibration, internal erosion, external corrosion, creep. Pressure cycle durability is one of the most important characteristic in the design of heavy duty radiator. Current design methodologies involve design of heavy duty radiator using the nominal finite element approach which does not take into account of the variations occurring in the geometry, material and boundary condition, leading to over conservative and uneconomical designs of radiator system. A new approach is presented in the paper to integrate traditional linear finite element method and probabilistic approach to design a heavy duty radiator by including the uncertainty in the computational model. As a first step, nominal run is performed with input design variables and desired responses are extracted. A probabilistic finite elementanalysis is performed to identify the robust designs and validated for reliability. Probabilistic finite element includes the uncertainty of the material thickness, dimensional and geometrical variation. Gaussian distribution is employed to define the random variation and uncertainty. Monte Carlo method is used to generate the random design points.Output response distributions of the random design points are post-processed using different statistical and probability technique to find the robust design. The above approach of systematic virtual modelling and analysis of the data helps to find efficient and reliable robust design.

Keywords: Heavy duty radiator, Nominal finite element analysis, Probabilistic finite element analysis, Monte Carlo method, Robust design, Reliability.

Abbreviations

AL3003	Aluminium Alloy 3003 Series
CAD	Computer Aided Design
CDF	Cumulative Distribution Function
DOF	Degree Of Freedom
EPDM	Ethylene Propylene Diene Monomer
FEA	Finite Element Analysis
MCM	Monte-Carlo Method
NAFEMS	National Agency for Finite Element Methods and Standards
PDF	Probabilistic Density Function
PSI	Pound Per Square Inch
THK	Thickness
YM	Young's Modulus

1. Introduction

In modern engine cooling system, radiator is the main heat exchanger, where the engine coolant rejects heat to the passing air and again passed to the water jacket to absorb some more heat from the engine. Design of the radiator is challenging due to higher operating pressure and limited packaging space. Beatenbough [1] found that pressure, thermal, road vibration, creep, internal erosion external corrosion are the failure modes during the life of a radiator. Out of which pressure cycle failure is found be one of the major contributor for the radiator field failures. Millard [2] defined the durability characteristics of the heavy duty radiator and found that the typical failure area is at the header-tube joint during pressure cycle loading. Eitel et al. [3] found a new concept of aluminium radiator and studied the pressure cycle durability characteristics. An outline of pressure cycle test requirement for the heavy duty commercial radiator is also proposed.

Nominal approach is based on the nominal dimension of the model and deterministic in nature. Nominal design leads to increased raw material usage, higher process time, reduced business profits. Traditionally nominal finite element analysis is performed to evaluate the structural performance of the radiator, which does not take into account of the variations occurring in the geometry, manufacturing tolerance, material properties, environmental loads etc. Macneal and Harder [4] introduced a series of problems to test the finite element result accuracy.

Capturing the uncertainty and variation in the design helps to reduce customer warranty claims and improve customer satisfaction. Probabilistic finite element approach helps to address the issues in the nominal analysis. Reh et al. [5] performed a probabilistic finite element analysis on a turbine blade by taking into account of variation of geometry, material and boundary condition. El-Sayed and Chassapis [6] enumerated a probabilistic finite element simulation approach to evaluate tooth root strength of the spur gears and explain the relationship between process variable uncertainties and performance. Berthaume et al. [7] showcased the application example of probabilistic finite element analysis on a human bone. Riha et al. [8] presented a procedure to perform probabilistic structural simulation using general purpose software.

Ahmad et al. [9] deployed Monte Carlo simulation technique to generate the random samples of the input functions for the probabilistic analysis. Guoliang et al.

[10] studied the efficiency of the Monte Carlo method and proves it as a best method to solve practical and complex engineering problems. Dar et al. [11] illustrated that no tolerance can be machined for infinitesimal tolerance or component material property is exactly known, rather represented by means of distribution functions. Also emphasize that the input design variables are not defined by single value but by a statistical distribution. Normal or Gaussian distribution is the most commonly used distribution function. Comer and Kjerengtroen [12] provided a basic introduction to common probability distribution, statistical terminology and MCM.

Kazmer and Roser [13] outlined a detailed approach to robust product design methods. Vlahinos and Kelkar [14] presented a weight reduction in body-in-white structure using a probabilistic approach, which utilizes a parametric deterministic FEA model. Burnside and Wu [15] indicated about the importance of cumulative distribution function used in the probabilistic structural analysis. Wu et al. [16] revealed the importance and benefits of cumulative distribution function by solving three application problems. Andreas and Kelkar [17] explained a practical example to understand the results of robust analysis. Thacker et al. [18] presented few application examples using commercial software. Patelli et al. [19] discussed about the advantage and limitations of the general purpose software for efficient un-certainty management. Stefanou [20] provided a state of art review of past and recent developments in the field of uncertainty modeling and analysis. The objective of the paper is to perform a probabilistic finite element analysis on a heavy duty radiator under pressure cycle loading to find a robust design which meets higher operating pressure and reliability targets.

2. Materials and Methods

2.1. Nominal approach

Problem definition is the first step of the nominal approach. In this study, internal pressure analysis of a heavy duty radiator is performed. CAD geometry of the radiator model is imported into a finite element modeling software. Hypermesh v.13is used for finite element modeling and optistruct v13 & hyperstudy v.13 [21] are used for the nominal run and probabilistic finite element analysis respectively.

Heavy duty truck radiator consists of a tube, header, fin, gasket, tank and core side [3] (Fig. 1). A heavy duty radiator CAD model is discretized using solid and shell elements. The finite element model consists of 86,774 nodes and 49566 elements. The finite elements are validated and verified by the NAFEMS benchmark [4]. To minimize the computational time, a quarter symmetry model of the radiator is developed for the nominal run. The finite element model includes the stiffness of tank, gasket, header, tube, fins and core side to simulate the exact physical behaviour of the radiator during the pressure cycling loading. Also the model takes into account of the corner tube joint which is critical part for the radiator pressure analysis [2]. Tank is modeled using second order tetrahedron elements for better accuracy. Gasket and fin are modelled using hex and penta elements. Tube, header and core side are modeled using shell elements. Finite element modelling errors can be minimised with right mesh size and pattern to avoid singularities and spurious stresses.

The core system including the tube, fins and core side are made from Al3003 [3]. The radiator plastic tank is an injection moulded from fiberglass-reinforced nylon pa66 and the gasket is made from epdm material [3]. Young's modulus of 69000 MPa and Possion's ratio of 0.33 is defined for the Al3003 and plastic tank materials. Fin geometry is modeled as block to reduce the model size and computational time. Equivalent orthotropic property is calculated and assigned for the fin blocks.

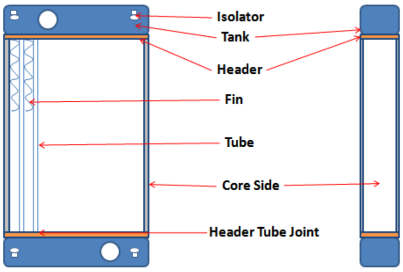

Fig. 1. Schematic diagram of commercial vehicle radiator.

2.2. Nominal approach boundary conditions

Due to the symmetry in the geometry, symmetrical boundary condition is assumed in the model (Fig. 2). The model is symmetrical about yz and xz plane. The normal axes x and y are constrained in the *yz* and *xz* symmetrical planes. The constraints are shown by green triangles in the Fig. 2. Pressure load is applied to the internal wetted surface of the tank, header and tubes. Flat faces of the isolators are constrained in all DOF. Typical pressure of 30*PSI* [3] is applied to the internal pressure surface shown as red arrows in Fig. 3. The operating pressure for the heavy duty commercial vehicle radiator is 15PSI. In the pressure cycle durability test of the radiator, the pressure amplitude will be considered as 1.5 to 2 times the operating pressure. In this study, pressure load is considered as30PSI.

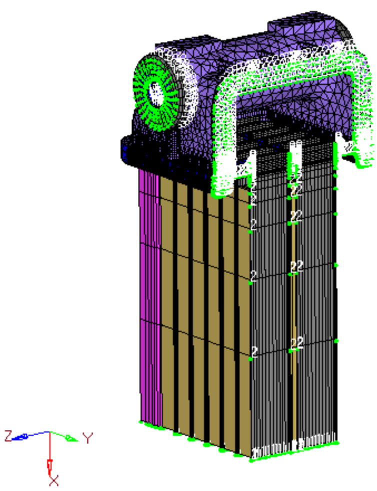

Fig. 2. Boundary condition.

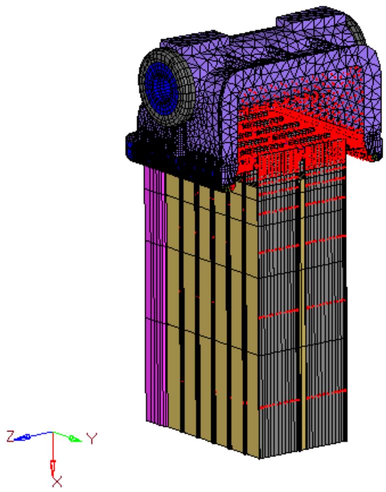

Fig. 3. Internal pressure load (30PSI).

2.3. Integratedapproach to robust design

The flowchart shown in Fig. 4 explains the integrated approach developed to combine nominal and probabilistic FEA to include the uncertainty in the design. The approachpredicts the reliability target for the nominal and probabilistic model and enables designers to identify robust, reliable and cost effective designs. Integrated approach helps to minimize the overdesign, excess material usage and variation in manufacturing process [13]. Also deploys various statistical and probabilistic tools to evaluate for a robust design.The finite element model size has been kept less than 100,000 nodes to study large sample size using the limited computational resource.

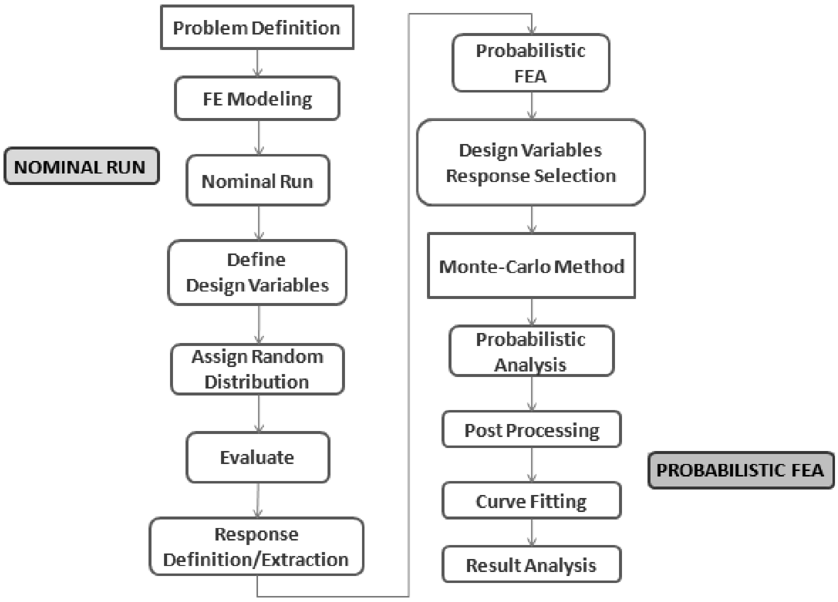

Fig. 4. Integrated robust design approach.

2.3.1. Design variable definition

Eight design variables (Table 1), impacting the pressure cycle life of the radiator are studied. Each of the variables is defined by a lower bound and upper bound values. The initial bound values of the input design variables are the nominal dimension of the current nominal design. The design variables are considered as continuous random variables [9].

The variation in the thickness is considered for the tube, core side and header components. Variation in the tube length is defined by the tube length variable. Tube length, tank profile design such as low profile (shorter height) and high profile (tall) are created using hyper morph functionality in the hyper mesh. These shape profiles are saved as shape variables. Remaining two variables are the variation in the material property (young’s modulus) for the tube and radiator tank material.

Table 1. Design variables summary.

Design Variable	Design Variable	Lower Bound	Initial Bound	Upper Bound
Tube Thickness	dv_1	0.3	0.4	0.5
Core Side Plate Thickness	dv_2	1	1.6	2
Header Thickness	dv_3	1	2	3
Tube Length	dv_4	-1	0	1
Low Profile Tank	dv_5	-1	0	1
High Profile Tank	dv_6	-1	0	1
Tank Young's Modulus	dv_7	2340	2600	2860
Tube Young's Modulus	dv_8	62100	69000	75900

2.3.2. Design variable distribution

The uncertainty of the design variable is defined by a normal distribution curve which is the function of the random variable [8]. Table 2 explains the distribution and statistical parameters used to represent the uncertainty in the design variable [9]. In the statistical approach, input parameters are randomized according to prescribed probabilistic distributions (Gaussian, log- normal, etc.) and a sampling algorithm. Each set of values for the random input parameters produces a set of results (i.e., displacement, and stress fields) through deterministic FEA. Post-processing of the results yields statistics for output variables such as deflection at specific nodes, maximum von Mises stress. The accuracy of output statistics can be improved with increased sampling [7].

Table 2. Design variable - Random distribution properties.

Design Variable	Design Variable	Distribution	Alpha	Beta
Tube Thickness	dv_1	Normal_Variance	0.4	0.0025
Core Side Plate Thickness	dv_2	Normal_Variance	1.6	0.0625
Header Thickness	dv_3	Normal_Variance	2	0.25
Tube Length	dv_4	Normal_Variance	0	0.25
Low Profile Tank	dv_5	Normal_Variance	0	0.25
High Profile Tank	dv_6	Normal_Variance	0	0.25
Tank Young's Modulus	dv_7	Normal_CoV	2600	0.1
Tube Young's Modulus	dv_8	Normal_CoV	69000	0.1

2.3.3. Random design generation

Monte Carlo method [10] is employed to generate the random design involving the eight design variable. For example, Fig. 5 shows that 250 random designs were generated for core side thickness design variable. Similarly random designs are generated for all the eight input design variables. MonteCarlomethod does not make any simplification or assumptions in the deterministic or probabilistic model [5]. With increasing number of samples, the MCM converges to the true and correct probabilistic result [5]. Max deformation and max von Mises stress are the two responses, studied for all the random design points.

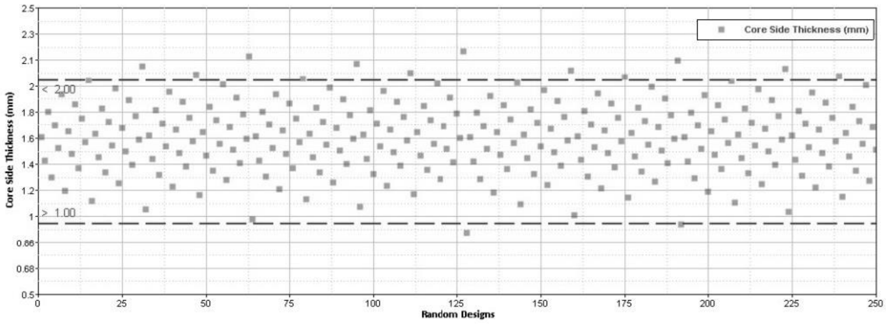

Fig. 5. Core side thickness random runs.

3. Results and Discussion

3.1. Nominal FEA results

Deformation plot of the radiator assembly shows maximum deformation at the core side due to the expansion of the core. Deformed and un-deformed state of the heavy duty radiator is shown in Fig. 6. The magnitude of the deformation is 0.12 mm. Max von Mises stress for the nominal run is 80 MPa (Fig. 7). Maximum stress location is at the header tube joint corner [2], which is typical high stress gradient for the pressure analysis. Maximum deformation and maximum von Mises stress are used as the output response function for the probabilistic FEA.

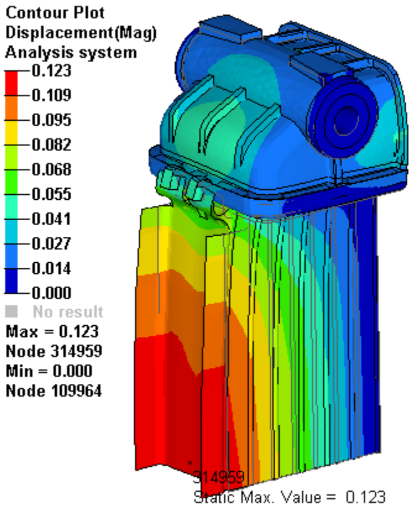

Fig. 6. Deformation plot (mm).

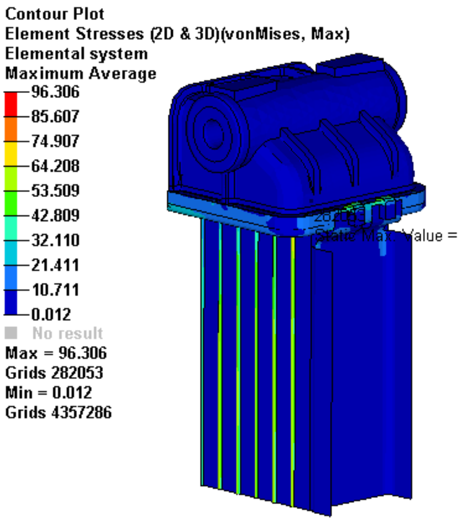

Fig. 7. Max stress plot (MPa).

3.2. Probabilistic FEA result discussion and analysis

The distribution of random design data are summarized in Table 3. The mean and median comparison shows that the entire design variable except the tube young's modulus is normally distributed and exhibits a symmetric distribution. Median of the tube young's modulus is greater than the mean of the tube young's modulus. Tube young's modulus design variable is having a negative skew and an asymmetric distribution. Standard deviation shows the dispersion of random data. Lower values of the standard deviation indicate the data points are close to the mean value of the distribution. Skewness values show the distribution shape and also indicate the number of mode (peak). Also suggest us whether the distribution is symmetric or skewed. As a rule of thumb, If the skewness value varies between -0.5 and +0.5, it is approximated as normal symmetric distribution. If skewness is less than −1 or greater than +1, the distribution is highly skewed. If skewness is between −1 and −½ or between +½ and +1, the distribution is considered moderately skewed. Analysis results show approximately normal distribution for all the input design variables and output response variables. Kurtosis defines the peakedness of the distribution, whether peak is tall/sharp or board/flat. Kurtosis indicates a flat distribution for all variables except the header thickness. A header thickness variable has normal distribution.Out of 250 random sampling points, two design runs failed due to skewed element quality during the profile generation.

Table 3. Statistical summary of the probabilistic simulation.

Design Variables	Mean	Variance	Skewness	Kurtosis
Tube Thickness	0.40	0.00	0.0	-0.2
Core Side Thickness	1.60	0.06	-0.1	-0.3
Header Thickness	1.99	0.24	-0.1	0.0
Header-Header	-0.01	0.25	-0.1	-0.1
Low Profile Tank	-0.02	0.24	-0.1	-0.2
High Profile Tank	0.00	0.23	0.0	-0.2
RAD Tank Modulus	2585	67373	-0.2	-0.1
Tube Modulus	68711	44500	-0.1	-0.3
Max Deformation	0.12	0.00	0.3	0.0
Max Stress	81	38.31	0.3	-0.2

3.2.1. Test for normality

To further validate the statistical data, various test for normality is performed, probabilistic value is calculated using the prominentknown methods and results are tabulated in Table 4. The calculated results show all the design variables exhibit normal distribution and random data points are normal.

Table 4. Test of normality results summary.

Normality Test Name	Tube Thickness	Core Side Thickness	Header Thickness	Low Tank Profile
Kolmogorov Smirnov	P>0.15	P>0.15	P>0.15	P>0.15
Anderson Darling	P = 1	P = 1	P = 1	P = 1
Lilliefors-Van Soest	P> 0.20	P> 0.20	P> 0.20	P> 0.20
Cramer von Mises	P = 1	P = 1	P = 1	P = 1
Ryan Joiner	P> 0.10	P> 0.10	P> 0.10	P> 0.10
D'Agostino Pearson	P = 0.759	P = 0.59	P = 0.786	P = 0.666
Sharpiro Wilks	P = 1	P = 0.991	P = 1	P = 0.997

If the test is not significant, the data are normal, p value greater than 0.05 indicates normality. If the test is significant, p value will be less than 0.01 that indicates the data are non-normal. To further validate the normality claim, we can plot a normal Q-Q plot. Q-Q stands for quantile &quantile plot or normal probability plot. If data points fall on line, data is considered normal, if does not fall on the line, then it is called non-normal. The normal Q-Q plot for the input design variables and output responses are plotted in Figs. 8 and 9. The p-values from the above table and normal Q-Q plot confirm a normally distributed random data points are used for the analysis.

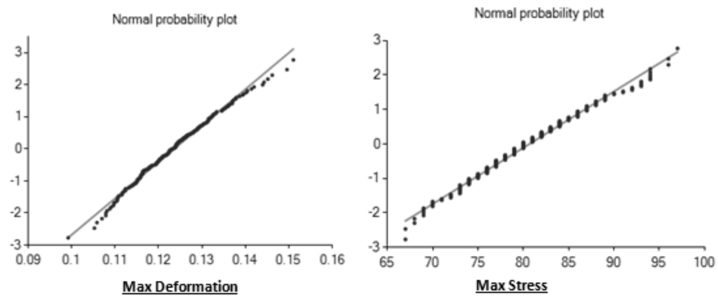

Fig. 8. Normal quantile-quantile of the output response variables.

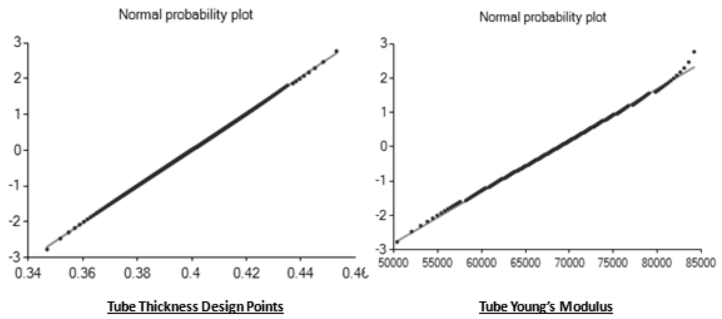

Fig. 9. Normal quantile-quantile of the input design variables.

3.2.2. Goodness of fit - response variables

It is important to check the goodness of fit for the output response calculated from 250 random design points.Goodness of fit is determined by the co-efficient of determination value (*R^2*). If the R^2 is close to the value 1, the model fit well with the observations. The *R^2*is tabulated for the max deformation (Table 5) and max stress response (Table 6). The table summary shows that both responses have got a good data fit for the 248 sampling points and variations are small.

Table 5. Deformation response error estimation.

Criterion	Value
***R*-Square**	0.99677
Relative Average Absolute Error	0.03946
Maximum Absolute Error	0.00233
Root Mean Square Error	5.16E-04

Table 6. Stress response error estimation.

Criterion	Value
***R*-Square**	0.99761
Relative Average Absolute Error	0.03466
Maximum Absolute Error	1.24502
Root Mean Square Error	3.02E-01
Number of Samples	248

3.2.3. PDF/CDF/Histogram plot - Input design variable

The histogram plot escribes the random variable distribution. A histogram is a graphical representation of the frequency distribution and consists of a series of rectangles where width represents an interval of the values of the random variables and height represents number of occurrences in the interval [12]. The probabilistic density function (PDF) [15] and cumulative distribution function (CDF) [14] for the tube thickness input design variable is shown in Fig. 10. PDF is a relative probability of the random variable, whereas CDF gives the absolute probability of the random variable. Monte Carlo method generates 250 random design points for all the input design variables. For example, tube thickness design variable, 173 design falls into 1 sigma level, 63 design falls in 2 sigma and 12 design falls in 3sigma level and confirms that the input design variables are normal distributed.

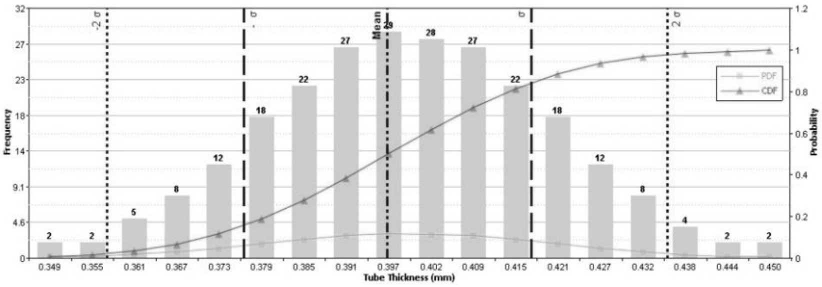

Fig. 10. Random distribution of the tube thickness input variable.

3.2.4. PDF/CDF/Histogram Plot - Output response function

PDF represents the probability of the variable having a given value. Area under the PDF is equal to one, representing all the possible occurrences [12].CDF is obtained by integrating the PDF from its lower limit to some value. CDF varies from 0 to 1.CDF [16] helps to identify the probability of failure for a given random variable interval. PDF and CDF plot for max deformation (Fig. 11) and max stress (Fig. 12) are shown below. Nominal displacement of 0.122 mm has 47% of probability to failure. Nominal stress of 80 MPa has 51% chances of probability of failure.

3.3. Reliability assessment

The reliability plot can be used to find unreliability and identify robust designs [17]. The graph in Fig. 13 shows the reliability vs. max deformation. The red triangles are 248 random designs generated using the MCM [20]. The black bold line is the deformation results of each random design. The results show that deformation equal to 0.11 mm has a reliability of 95% and deformation greater than 0.14 mm has 5%

reliability. The nominal deformation of 0.122 mm has a reliability of 54%. Similarly for the max stress equal to 70.6 MPa has 95% reliability and max stress greater than 92.41 MPa has 5% reliability respectively (Fig. 14). Whereas nominal stress value of 80.4 MPa has 49% reliability. Therefore nominal design is having 50% reliability and 50% unreliability leading to improper designs.

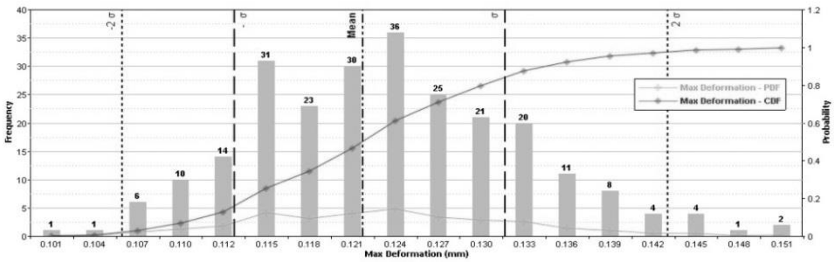

Fig. 11. Random distribution of the max deformation response.

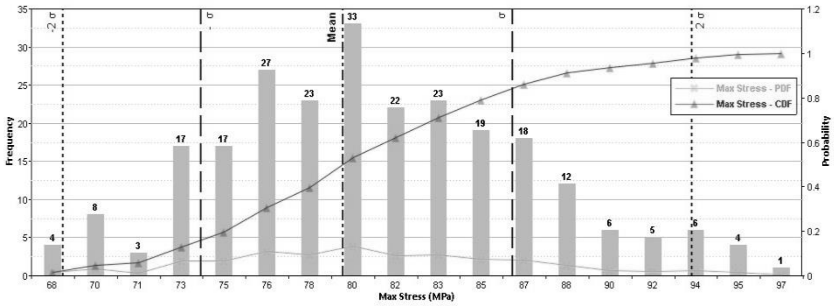

Fig. 12. Random distribution of the max stress response.

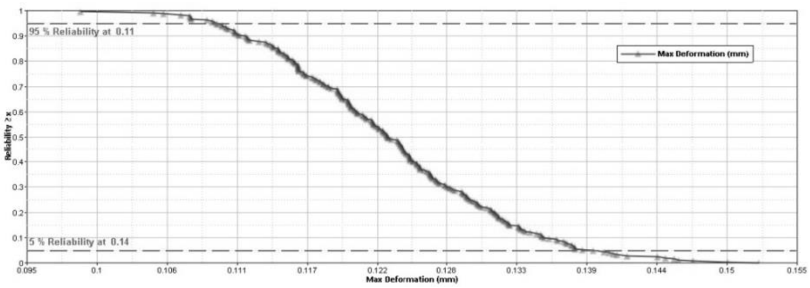

Fig. 13. Reliability plot of max deformation (mm).

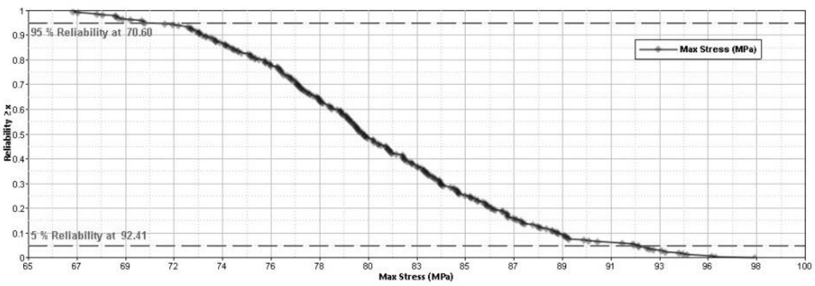

Fig. 14. Reliability plot of max stress (MPa).

3.4. Correlation study

Correlation matrix explains the sensitivity [8] between the input design variables and output response function and identifies the most significant variable [19]. The Table 7 shows that tube thickness has a prefect negative relationship with the max stress. Tube thickness also has a strong negative relation with max deformation. Tube young's modulus has a moderate negative correlation with max deformation. Remaining relationship between the design variable and responses function are in-significant. Colour coding key in Table 8 is provided to understand the correlation matrix.

Table 7. Correlation matrix: Output response vs. input design.

	Max Deformation	Max Stress	Tube Thk	Core side Thk	Header Thk	Header Header
Max Deformation	1.0	0.6	-0.8	-0.1	0.0	0.2
Max Stress	0.65	1.0	-1.0	0.0	-0.1	0.0
Tube Thk	-0.8	-1.0	1.0	0.0	0.0	0.0
Core side Thk	-0.1	0.0	0.0	1.0	0.0	0.0
Header Thk	0.0	-0.1	0.0	0.0	1.0	0.0
Header - Header	0.2	0.0	0.0	0.0	0.0	1.0

+ Symbol	**Positive correlation**
- Symbol	**Negative correlation**

Table 8. Correlation matrix - Coding key.

Magnitude	Strength of the Relationship
1.0	prefect
0.8 to 1.0	very strong
0.6 to 0.8	strong
0.4 to 0.6	moderate
0.2 to 0.4	weak
0.0 to 0.2	none to extremely weak

Table 9. Reliability summary - Stress response.

Response	Bound Value (MPa)	Reliability	Probability of Failure	Comment
Max Stress (r_2)	67	99%	1%	Robust Design
Max Stress (r_2)	70	96%	4%	Target Design
Max Stress (r_2)	80	51%	49%	Nominal Design

Table 10. Reliability summary - Deformation response.

Response	Bound Value (mm)	Reliability	Probability of Failure	Comment
Max Deformation (r_1)	0.099	99%	1%	Robust Design
Max Deformation (r_1)	0.107	98%	2%	Target Design
Max Deformation (r_1)	0.123	53%	47%	Nominal Design

Table 11. Robust design summary.

Design #	Tube Thk (mm)	Core Side Thk (mm)	Header Thk (mm)	Max Deformation (mm)	Max Stress (MPa)	Reliability (%)
243	0.437	1.8	0.5	0.112	67.00	99.6%
250	0.453	1.5	2.1	0.099	67.24	99.2%
248	0.445	1.3	2.4	0.110	68.09	98.8%
249	0.448	1.7	1.6	0.107	68.35	98.4%
247	0.443	2.0	1.9	0.108	68.93	98.0%
238	0.433	1.6	2.1	0.125	68.94	97.6%
239	0.433	2.0	2.6	0.120	69.08	97.2%
245	0.440	1.7	2.2	0.109	69.25	96.8%
241	0.435	1.6	2.2	0.115	69.61	96.4%
240	0.434	1.2	1.8	0.118	70.10	96.0%
244	0.438	1.4	1.8	0.114	70.18	95.6%
246	0.441	1.6	1.4	0.109	70.24	95.2%
221	0.424	1.7	2.4	0.124	71.14	94.8%

4. Conclusion.

Nominal max stress is identified as 80 MPa at the header-tube joint, which has reliability of 49%. Nominal max deformation is at 0.122 mm at the core sides are at 50% reliability. The nominal reliability % indicates that the nominal finite element analysis does not factor in the uncertainty and variations. The integrated robust design approach enables the product designer to identify robust designs which takes into account of variation arising to geometry, manufacturing process, loading, material property, etc. Robust designs are less sensitive to variation and results in meeting reliability targets. Robust design gives max stress of 67 MPa with 99% reliability (Table 9) and max deformation of 0.09 mm at 99% reliability (Table 10). The correlation study shows that tube thickness has strong negative correlation with maximum stress and max deformation.Tube young's modulus has a moderate negative correlation with the max deformation.

The probabilistic finite element analysis has resulted in 13 robust designs with a reliability of greater than 95%. The robust designs with design variable dimension are tabulated in the Table 11. Future scope can include effect of road vibration loading in the nominal design. Reliability of the robust can be further maximized by performing a reliability based design optimization. Combining nominal finite element analysis with probabilistic finite element analysis enables the designers and engineers to figure out a robust design due uncertainty in the model formulation and input parameters.

References

1. Beatenbough, P.K. (1992). Radiator design and material recommendation based on failure mode analysis. *Society of Automotive Engineers*, Paper No. 920178.
2. Millard, G.S.T. (1977) Analysis of durability characteristics of heavy duty radiators. *Society of Automotive Engineers*, Paper No.770024.
3. Eitel, J.; Woerner, G.T.; Horoho, S.; and Mamber, O. (1999). The aluminium radiator for heavy duty trucks. *Society of Automotive Engineers*, Paper No 1999-01-3721.
4. Macneal, R.H.; and Harder, R.L. (1985). A proposed standard set of problems to test finite element accuracy. *Finite Elements in Analysis and Design*, 1(1), 3-20.
5. Reh, S.; Beley, J.-D.; Mukherjee, S.; and Khor, E.H. (2006). Probabilistic finite element analysis using ANSYS. *Structural Safety*, 28(1-2), 17-43.
6. El-Sayed, S.; and Chassapis, C. (2011). Probabilistic simulation approach to evaluate the tooth-root strength of spur gears with FEM-based verification. *Engineering*, 3(1), 1137-1148.
7. Berthaume, M.A.; Dechow, P.C.; Iriarte-Diazd, J.; Ross, C.F.; Strait, D.S.; Wang, Q.; and Grosse, I.R. (2012). Probabilistic finite element analysis of a craniofacial finite element model. *Journal of Theoretical Biology*, 300, 242-253.
8. Riha, D.S.; Millwater, H.R; and Thacker, B.H. (1992). Probabilistic structural analysis using a general purpose finite element program. *Finite Elements in Analysis and Design*, 11(1), 201-211.
9. Ahmad, Z.; Ariffin, A.K.; and Rahman, M.M. (2011). Probabilistic finite element analysis of vertebrae of lumbar spine under hyperextension loading. *International Journal of Automotive and Mechanical Engineering*, 3(1), 256-264.
10. Guoliang, J.; Lin, C.; and Jiamei, D. (1993). Monte Carlo finite element method of structure reliability analysis. *Reliability Engineering and System Safety*, 40(1), 77-83.
11. Dar, F.H.; Meakin, J.R.; and Aspden, R.M. (2002). Statistical methods in finite element analysis. *Journal of Biomechanics*, 35(9), 1155-1161.
12. Comer, J.; and Kjerengtroen, L. (1996). Probabilistic methods in design introductory topics. *Society of Automotive Engineers*, Paper No. 961792.
13. Kazmer, D.; and Roser, C. (1999). Evaluation of product and process design robustness. *Research in Engineering Design*, 11(1), 20-30.

14. Vlahinos, A.; and Kelkar, S. (2001). Body-in-white weight reduction via probabilistic modeling of manufacturing variations. *Society of Automotive Engineers*, Paper No. 2001-01-3044.
15. Burnside, O.H.; and Wu, Y.-T. (1988). Probabilistic structural analysis methods. *Society of Automotive Engineers*, Paper No. 880784.
16. Wu, Y.-T.; Millwater, H.R.; and Cruse, T.A.(1990). Advanced probabilistic structural analysis methods for implicit performance functions. *The American Institute of Aeronautics and Astronautics AIAA Journal*, 28(9), 1663-1669.
17. Vlahinos, A.; and Kelkar, S. (2002). Designing for Six Sigma quality with robust optimization using CAE. *Society of Automotive Engineers*, Paper No 2002-01-2017.
18. Thacker, B.H.; Riha, D.S.; Fitch, S.H.K.; Huyse, L.J.; and Pleming, J.B. (2006). Probabilistic engineering analysis using the NESSUS software. *Structural Safety*, 28(1-2), 83-107.
19. Patelli, E.; Panayirci, H.M.; Broggi, M.; Goller, B.; Beaurepaire, P.; Pradlwarter, H.J.; and Schuëller, G.I. (2012). General purpose software for efficient uncertainty management of large finite element models. *Finite Elements in Analysis and Design*, 51, 31-48.
20. Stefanou, G. (2009). The stochastic finite element method: Past, present and future. *Computer Methods in Applied Mechanics and Engineering*. 198(9-12),1031-1051.
21. Altair® HyperWorks, User Guide, Version 13, Altair Engineering Inc.2016.