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ROCHE-COORDINATES AND METRIC COEFFICIENTS FOR DIFFERENTIALLY ROTATING POLYTROPIC STARS

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Abstract

Kopal introduced a new system of coordinates, to study the problems of close binary stars, called Roche-coordinates. Besides studying the other problems of close binary stars, Kopal also considered the possibility of using this system of coordinates to study the problem of stability and oscillations of rotating stars in close binary systems. This paper insists that the mathematical modeling may be applicable for the problems about the stars. Kopal obtained explicit expressions of three curvilinear coordinates for the system of Roche-coordinates associated with the Roche-model of a star, distorted by solid body rotational forces. One coordinate can be defined in by the equipotential surface, which is known in close algebraic form. Second becomes identical with the meridional planes of rotationally distorted model, while the third follows from the requirements of the orthogonality to the others. The expressions of these coordinates in terms of polar or cartesian system, but in the form of infinite series are converge rapidly for any distortion below that which is entail equatorial break-up. Following Kopal's approach, we obtained the explicit expressions for Roche-coordinates associated with the polytropic models of stars distorted by differential rotation. These coordinates and harmonics can be used to define the structure and local/vibrational stability of the polytropic Roche-models.

Keywords: Differential rotation, Tidal distortion, Roche-equipotential, Equilibrium-structure, Roche-limit, Close binary system, Metric-coefficient.

1. Introduction

Kopal [1], Kopal and Kitamura [2] introduced a new system of coordinates to study the problems of close binary stars called Roche-coordinates. Kopal [1] also considered the possibility of using this system of coordinates to study the problems of stability and oscillations of rotating stars in close binary system. To

Nomenclatures

a_i, b_i	Constants
b_1, b_2, b_3	Constants
G	Centre of gravity
h_1, h_2, h_3	Metric coefficients
M, M'	Mass for primary and secondary
N	Polytropic indices
V	Gravitational potential
R	Point of separation, distances
r, r^1	Distances
s	Modified non-dimensional cylindrical coordinate
x, y, z	Points spherical coordinates
Greek Symbols	
ξ, η, ζ	Roche-Coordinates
Ω	Total potentials
ω	Angular velocity of the rotation

extend this analysis Faye assumed a law of differential rotation of the type $\omega=b_1+b_2s^2$ (where ω *is* the angular velocity of rotation of a fluid element at a distance s from the axis of rotation, while b_1 and b_2 are certain constants) to account the effect of differential rotation of the sun. Woodard [3] considered a law of differential rotation of the type $\Omega(x)=B_0+B_1x^2+B_2x^4$, where Ω *is* an even function of latitude$x$. Geroyannis and Antonakopoulas [4] studied the structural distortion of polytropic stars by differential rotation using a law of differential rotation earlier proposed by Clement [5]. According to this law, the angular velocity $\omega(s)$ of a fluid element is given by $\omega(s)=\sqrt{\sum_{i=1}^{3} a_i e^{-b_i s^2}}$, where, s is a modified non-dimensional cylindrical coordinate's a_i and b_i are constants. Clement [5] has given the values of parameters a_i and b_i for the various polytropic indices. This law is used by Kumar et al. [6] to determine the equilibrium structure of differentially rotating and tidally distorted polytropic stars.Table 1 shows the collected data is taken from Clement [5].

Mohan and Singh [7, 8] and Saini et al.[9], have used the law of differential rotation of the form $\omega=b_1+b_2s^2$ while, Kumar et al. [10] have used the law of differential rotation of the form $\omega=b_1+b_2s^2+b_3s^4$ to obtain expressions for the Roche-coordinates associated with the Roche-model of a star distorted by differential rotation. Lal et al. [11] have checked the validity of series expansion being used for the position of a point on a Roche-equipotential However, for all case, the higher terms in s are not to be considered. The study becomes more complex if s is included in higher ordered terms.

In this paper, we have obtained the Roche-coordinates and Roche-harmonics for polytropic models using the law of differential rotation in the form $\omega(s)=\sqrt{\sum_{i=1}^{3} a_i e^{-b_i s^2}}$, where 'ω' is the angular velocity of rotation.

Table 1. Collected Data from Clement [5].

Values of a_i and b_i	N=2.00	N=2.50	N=3.00	N=3.25
a_1	+0.546668	+0.263144	+0.095155	+0.048836
a_2	+0.544726	+0.720053	+0.555735	+0.400167
a_3	+0.091395	+0.016858	+0.350959	+0.550992
b_1	+0.117936	+0.097485	+0.051248	+0.037318
b_2	+0.38444	+0.290017	+0.203307	+0.153630
b_3	+0.714485	+0.21676	+0.594146	+0.490194

2. System of Roche-Coordinates

To study the problems of distorted stars, Kopal [1,12] introduced the system of Roche-coordinates m, in which the equipotential surfaces of a distorted Roche-model are chosen to represent the equipotential surfaces of an actual stellar model distorted by rotational and tidal forces. This system is used by Kopal [13] to determine the vibrational stability of rotating stars.

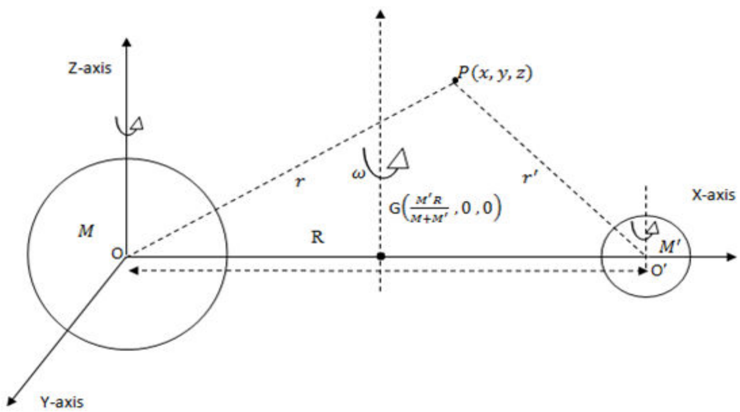

Fig. 1. Axes of Reference.

In our study let M and M' be the total mass of the primary and secondary components of a binary system which are gaseous spheres. Let R be the point of separation between centres of these two masses. This binary system is called rectangular system of Cartesian-coordinates having the origin at the centre of gravity of mass M. The x-axis is along the line joining the centres of two components and z-axis perpendicular to the plane of the orbit.

In this system of coordinates, the centre of gravity G may be written as $\left[\frac{M'R}{M+M'},0,0\right]$ and the total potential Ω at an arbitrary point $P(x, y, z)$ of the combined forces of gravitation and rotation is given by

$$\Omega = G\frac{M}{r} + G\frac{M'}{r'} + \frac{\omega^2}{2}\left[\left(x - \frac{M'R}{M+M'}\right)^2 + y^2\right] \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$ and $r'^2 = (R-x)^2 + y^2 + z^2$ while r and r' represent the distances of point p from the centre of gravity, the total potential Ω is the sum of potential arising from the mass of the component of mass M, disturbing potential of its companion of mass M' and potential arising from the centrifugal force.

The angular velocity ω is identical with the Keplarian angular velocity in close binary system so that

$$\omega^2 = G\frac{M+M'}{R^3} \tag{2}$$

If we insert above both relations (1) and (2) and adopt as M our unit mass, R as unit of length and choose the unit of time such that G =1, Equation (1) may be expressed in terms of polar spherical coordinates

$$x = r\cos\phi\sin\theta = r\lambda\,,\ \ y = r\sin\phi\sin\theta = r\mu\ \ ,\ z = r\cos\theta = r\nu \tag{3}$$

as $$\xi = \frac{1}{r} + q\left[\frac{1}{\left(1-2\lambda r + r^2\right)^{\frac{1}{2}}} - \lambda r\right] + \frac{1}{2}\omega^2 r^2\left(1-\nu^2\right), \tag{4}$$

where, $\xi = \frac{R\Omega}{GM} - \left[\frac{M'^2}{2M\left(M+M'\right)}\right]$ and $q = \frac{M'}{M}$ are non-dimensional parameters and ω^2 is non-dimensional unit of GM/R^3.

The left hand side of Eq. (4) represents those surfaces which are generated by setting ξ constant are referred to as Roche-equipotentials.The form of Roche-equipotential depends entirely upon the value of ξ. If ξ is large, the corresponding equipotentials will consist of two separate ovals, closed around each of the two mass points (see Fig. 1) The value of right hand side of Eq. (4) can be large only if r or r' becomes small.Therefore, large value of ξ corresponds to equipotentials, which differ but little from spheres surrounding one of the two mass centres. Both ovals will unite in a single point on the x-axis to form a dumbbell like configuration. These limiting values of ξ are called Roche-limits. In the system of Roche-coordinates {ξ, η, ζ,} we take the ξ coordinate to be an equipotential surface of the form(1) and choose the other two coordinates η and ζ in such a way as to satisfy the conditions of mutual orthogonality with respect to ζ as well as each other.

The general equations orthogonality, which must be satisfied by any curvilinear system of coordinates are

$$\left.\begin{aligned} \xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z &= 0 \\ \xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z &= 0 \\ \eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z &= 0 \end{aligned}\right\}$$
(5)

where suffixes x, y, z represent the partial differentiation with respect to x, y and z. The transformation

$$(dx)^2 + (dy)^2 + (dz)^2 = h_1^2(d\xi)^2 + h_2^2(d\eta)^2 + h_3^2(d\zeta)^2 \tag{6}$$

of the metric element is specified by the metric coefficients.

$$h_1^{-2} = \xi_x^2 + \xi_y^2 + \xi_z^2, h_2^{-2} = \eta_x^2 + \eta_y^2 + \eta_z^2, h_3^{-2} = \zeta_x^2 + \zeta_y^2 + \zeta_z^2 \tag{7}$$

The direction cosines of a normal to the surface ξ = constant, η=constant and ζ=constant are given respectively by the ratios

$$\left.\begin{array}{lll} l_1 = h_1\xi_x, & m_1 = h_1\xi_y, & n_1 = h_1\xi_z \\ l_2 = h_2\eta_x, & m_2 = h_2\eta_y, & n_2 = h_2\eta_z \\ l_3 = h_3\zeta_x, & m_3 = h_3\zeta_y, & n_3 = h_3\zeta_z \end{array}\right\} \tag{8}$$

The above equations hold good for any triple-orthogonal curvi-linear system of coordinates. Kopal [14] and his team found the mathematical properties of this system of Roche-coordinates. This approach tells us that it is not possible, in general to obtain the expressions for η and ζ in closed analytic forms. Kopal and Sekender [15] and there after Singh [16], Singh and Gupta [17] found out two particular cases of this problem in detail. The first case corresponds to Roche-coordinates of a star distorted by rotational forces alone by putting m'=0 or $q=0$ and second case corresponds to Roche-coordinates of non-rotating star distorted by the tidal effects of a companion star by putting $w=q=1$.

3. Explicit Expressions of Roche-Coordinates for Roche-Model of a Star Distorted by Differential Rotation

We assume the Roche-model of a star of mass m, rotating according to the law

$$\omega^2 = \sum_{i=1}^{3} a_i \, exp\left(-b_i s^2\right) \tag{9}$$

The equation of hydrostatic equilibrium may be written in the form

$d\Omega = dV + \omega^2 ds^2$ or

$$d\Omega = dV + \omega^2(s)ds \quad \text{or} \quad \Omega = V + \int \omega^2(s) s ds \tag{10}$$

where Ω represents the total potential at a point p at distance r from the centre of the star, and $V = \dfrac{Gm}{r}$ is the gravitational potential, ω is angular velocity of rotation of an element of the star at distance s from the axis of rotation.

On using Eq. (9) into Eq. (10) and integrating, we get

$$\Omega = V + \frac{1}{2}\sum_{i=1}^{3}\frac{a_i}{b_i}\left(1 - e^{-b_i s^2}\right), \qquad \text{where } s^2 = x^2 + y^2 = r^2\left(1 - \nu^2\right)$$

$$= \frac{Gm}{r} + \frac{1}{2}\sum_{i=1}^{3}\frac{a_i}{b_i}\left(1 - e^{-b_i\left(x^2+y^2\right)}\right)$$

The non-dimensional form of the above equation can be expressed as

$$\xi = \frac{1}{r} + \frac{1}{2}\sum_{i=1}^{3}\frac{a_i}{b_i}\left\{1 - e^{-b_i r^2\left(1-\nu^2\right)}\right\}$$

$$= \frac{1}{r} + \frac{1}{2}\Sigma\left[a_i r^2 \left(1-\nu^2\right) - \frac{1}{2} a_i b_i r^4 \left(1-\nu^2\right)^2 + \frac{1}{6} a_i b_i^2 r^6 \left(1-\nu^2\right)^3 - \right.$$

$$\left. \frac{1}{24} a_i b_i^3 r^8 \left(1-\nu^2\right)^4 + .. \right] . \tag{11}$$

In the absence of rotation $a_1=a_2=a_3=0$, i.e., $a_i=0$ the Roche-equipotential (11) reduces to ξ, which is the potential of a star having solid body rotation. First we take $r_o = 1/\xi$ as our first approximation to the distance of the equipotential surface from the centre, then we can take second approximation as $r_1 = r_0 + \Delta r = r_0\left(r_0 + \frac{\Delta r}{r_0}\right)$.

Substituting the value of r_1 for r in Eq. (11) , we get

$$r = r_0\left[1 + \frac{r_0}{2}\sum_{i=1}^{3}\frac{a_i}{b_i}\left\{1 - e^{-b_i r_0^2\left(1-\nu^2\right)}\right\}\right]. \tag{12}$$

Above expression is correct up to second-order terms in angular velocity ω. Now we shall introduce the system of Roche-coordinates ξ, η, and, ζ, in which ξ coordinate defined by Roche-equipotential surface is given in closed algebraic form by Eq. (11), while the coordinates η and ζ are defined by their requirement that they are orthogonal to ξ as well as with respect to each other. The conditions which must be fulfilled by any system of orthogonal curvilinear coordinates are same as given in Eq. (5).

On differentiating Eq. (11) partially with respect to r, λ, and υ and neglecting powers of angular velocity greater than two, we get

$$\xi_r = -\frac{1}{r^2} + r\left(1-\nu^2\right)\omega^2 , \xi_\lambda = 0 , \xi_\nu = -r^2\nu\omega^2 . \tag{13}$$

By transformation of coordinates, it follows that

$$\xi_x = -\frac{\lambda}{r^2} + \lambda r\omega^2 , \xi_y = -\frac{\mu}{r^2} + \mu r\omega^2 , \xi_z = -\frac{\nu}{r^2} \tag{14}$$

and,

$$\left(\xi_x^2 + \xi_y^2 + \xi_z^2\right)^{-\frac{1}{2}} = r^2\left[1 + r^3\left(1-\nu^2\right)\omega^2 + ...\right]. \tag{15}$$

Now by the Lagrange’s differential equation

$$\frac{dx}{\xi_x} = \frac{dy}{\xi_y} + \frac{dz}{\xi_z}. \tag{16}$$

The Roche-coordinates η, ζ corresponding (in limit) to the angular coordinates to the spherical polars, will be obtained after integrating Eq. (16).

Dividing Eq. (16) by the normal element ds for a suitable integration

$$\frac{dy/ds}{dx/ds} = \frac{\xi_y}{\xi_x} \text{ and } \frac{dz/ds}{dx/ds} = \frac{\xi_z}{\xi_x}, \tag{17}$$

where, $\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}$ are the direction cosines of a vector normal to the surface ξ = constant. Therefore, by the condition of direction cosines, we have

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1. \tag{18}$$

By Eqs. (17) and (18),

$$\frac{dx}{ds}=\frac{-\xi_x}{\left(\xi_x^2+\xi_y^2+\xi_z^2\right)^{\frac{1}{2}}},\frac{dy}{ds}=\frac{-\xi_y}{\left(\xi_x^2+\xi_y^2+\xi_z^2\right)^{\frac{1}{2}}},\frac{dz}{ds}=\frac{-\xi_z}{\left(\xi_x^2+\xi_y^2+\xi_z^2\right)^{\frac{1}{2}}}. \tag{19}$$

The negative sign of the square root being taken because $\xi(x\ ,y,\ z)$ is a diminishing function of s . Now from Eq. (3)

$$\frac{dx}{ds}=r\frac{d\lambda}{dr}+\lambda,\frac{dy}{ds}=r\frac{d\mu}{dr}+\mu,\frac{dz}{ds}=r\frac{d\nu}{dr}+\nu\ . \tag{20}$$

Replacing ds by line element dr within our scheme of first-order approximation because ds is very small so from (14), (15) and (19), we have

$$\left.\begin{aligned}\frac{dx}{dr}&=\lambda\left[1-\omega^2r^3\nu^2-\omega^4r^6\left(1-\nu^2\right)\right],\\ \frac{dy}{dr}&=\mu\left[1-\omega^2r^3\nu^2-\omega^4r^6\left(1-\nu^2\right)\right]\\ \frac{dz}{dr}&=\lambda\left[1+\omega^2r^3\left(1-\nu^2\right)\right].\end{aligned}\right\} \tag{21}$$

Now from (20) and (21) we get

$$\frac{d\lambda}{dr}=-\omega^2r^2\nu^2\lambda\ ,\ \frac{d\mu}{dr}=-\omega^2r^2\nu^2\mu\ ,\ \frac{d\nu}{dr}=\omega^2r^2\nu\left(1-\nu^2\right). \tag{22}$$

On integrating Eq. (22) by Picard's method of successive approximation up to first approximation, we get

$$\begin{aligned}\lambda_1=-\lambda_0\nu_0^2\Bigg[&\sum_{i=1}^{3}a_i\frac{r^3}{3}-\sum_{i=1}^{3}a_ib_i\frac{r^5\left(1-\nu_0^2\right)}{5}+\sum_{i=1}^{3}a_ib_i^2\frac{r^7\left(1-\nu_0^2\right)^2}{14}\\ &-\sum_{i=1}^{3}a_ib_i^3\frac{r^9\left(1-\nu_0^2\right)^3}{54}+\sum_{i=1}^{3}a_ib_i^4\frac{r^{11}\left(1-\nu_0^2\right)^4}{264}+\ldots\Bigg]+c_1\end{aligned} \tag{23}$$

$$\begin{aligned}\mu_1=-\mu_0\nu_0^2\Bigg[&\sum_{i=1}^{3}a_i\frac{r^3}{3}-\sum_{i=1}^{3}a_ib_i\frac{r^5\left(1-\nu_0^2\right)}{5}+\sum_{i=1}^{3}a_ib_i^2\frac{r^7\left(1-\nu_0^2\right)^2}{14}\\ &-\sum_{i=1}^{3}a_ib_i^3\frac{r^9\left(1-\nu_0^2\right)^3}{54}+\sum_{i=1}^{3}a_ib_i^4\frac{r^{11}\left(1-\nu_0^2\right)^4}{264}+\ldots\Bigg]+c_2\end{aligned} \tag{24}$$

$$\nu_1=-\nu_0\left(1-\nu_0^2\right)\Bigg[\sum_{i=1}^{3}a_i\frac{r^3}{3}-\sum_{i=1}^{3}a_ib_i\frac{r^5\left(1-\nu_0^2\right)}{5}+\sum_{i=1}^{3}a_ib_i^2\frac{r^7\left(1-\nu_0^2\right)^2}{14}$$

$$-\sum_{i=1}^{3} a_i b_i^3 \frac{r^9\left(1-\nu_0^2\right)^3}{54} + \sum_{i=1}^{3} a_i b_i^4 \frac{r^{11}\left(1-\nu_0^2\right)^4}{264} + \ldots \Bigg] + c_3 \tag{25}$$

where c_1, c_2 and c_3 are integration constants and enforcing the condition $\lambda_1^2+\mu_1^2+\nu_1^2=1$, it follows that $c_1=\lambda_0$, $c_2=\mu_0$ and $c_3=\nu_0$, where zero subscript refers to zeroth approximation. In polar coordinate λ_0, μ_0 and ν_0 are given by $\lambda_0 = \cos\phi\sin\theta, \mu_0 = \sin\phi\sin\theta, \nu_0 = \cos\theta$, also for zeroth approximation $\lambda_0^2+\mu_0^2+\nu_0^2=1$. Therefore, using dependence of λ_0, μ_0 and ν_0 on η and ζ, we propose to use the following substitution in case of rational distortion as;

$$\lambda_1 = \cos\eta\sin\zeta\ ,\ \mu_1 = \sin\eta\sin\zeta\ ,\ \nu_1 = \cos\zeta \tag{26}$$

Using Eqs. (25) and (26), we have

$$\cos\eta = \lambda_1 / \left(1-\nu_1^2\right)^{\frac{1}{2}} \tag{27}$$

$$\cos\zeta = \nu_0\left[1 - \frac{\left(1-\nu_0^2\right)}{83160} r^3 \left\{ 27720\sum_{i=1}^{3} a_i - 16632\sum_{i=1}^{3} a_i b_i r^2\left(1-\nu_0^2\right) \right.\right.$$
$$+ 5940\sum_{i=1}^{3} a_i b_i^2 r^4\left(1-\nu_0^2\right)^2 - 1540\sum_{i=1}^{3} a_i b_i^3 r^6\left(1-\nu_0^2\right)^3$$
$$\left.\left. + 315\sum_{i=1}^{3} a_i b_i^4 r^8\left(1-\nu_0^2\right)^4 - \ldots \right\}\right], \tag{28}$$

whereas, the expression for η given in (27) is exact, the expression of ζ obtained in (28) is correct up to second ordered terms in angular velocity.

4. The Metric Coefficients

In this section, the explicit expressions of metric coefficient h_1, h_2 and h_3 associated with the Roche-coordinates (ξ, η, ζ) for the Roche-model of a star distorted by differential rotation are presented. The metric coefficients are obtained.

On using Eq. (7), we have

$$h_1 = r_0^2 + 2\sum_{i=1}^{3} a_i r_0^5 (1-\nu_0^2) - \frac{3}{2}\sum_{i=1}^{3} a_i b_i r_0^7 (1-\nu_0^2)^2 + \left(\frac{1}{4}\sum_{i=1}^{3} a_i + \frac{1}{2}\sum_{i=1}^{3} a_i a_j\right.$$
$$\left. + \frac{5}{2}\sum_{i=1}^{3}\sum_{k=1}^{3} a_i a_k \right) r_0^8 (1-\nu_0^2)^2 + \frac{2}{3}\sum_{i=1}^{3} a_i b_i^2 r_0^9 (1-\nu_0^2)^3$$
$$-4\sum_{i=1}^{3}\sum_{k=1}^{3} a_i a_k b_k r_0^{10} (1-\nu_0^2)^3 + \left(\frac{5}{2}\sum_{i=1}^{3}\sum_{k=1}^{3} a_i a_k + 5\sum_{i,j=1}^{3}\sum_{k=1}^{3} a_i a_j a_k\right) r_0^{11} (1-\nu_0^2)^3$$
$$-\frac{5}{24}\sum_{i=1}^{3}\sum_{k=1}^{3} a_i b_i^3 r_0^{11} (1-\nu_0^2)^4 + \left(\frac{1}{16}\sum_{i=1}^{3} a_i^2 b_i^2 + \frac{11}{8}\sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j b_j\right.$$

$$+\frac{11}{6}\sum_{i=1}^{3}\sum_{k=1}^{3}a_ia_kb_k^2\Bigg)r_0^{12}(1-\nu_0^2)^4-\frac{5}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}a_ia_jb_ja_kr_0^{13}(1-\nu_0^2)^3$$

$$-\left(\frac{5}{2}\sum_{i=1}^{3}\sum_{k=1}^{3}a_ia_kb_k+5\sum_{i=1}^{3}\sum_{j,k=1}^{3}a_ia_ja_kb_k\right)r_0^{13}(1-\nu_0^2)^3+\Bigg(\frac{5}{4}\sum_{i=1}^{3}\sum_{k=1}^{3}a_i^3a_k$$

$$+\frac{15}{4}\sum_{i=1}^{3}\sum_{k=1}^{3}a_ia_j^3a_k+\frac{15}{2}a_1a_2a_3\sum_{i}^{3}a_i\Bigg)r_0^{14}\left(1-\nu_0^2\right)^4$$

$$+\frac{5}{4}\sum_{i=1}^{3}\sum_{k=1}^{3}a_ia_kb_k^2r_0^{15}\left(1-\nu_0^2\right)^5+\ldots \tag{29}$$

Proceeding like manner consistent with our scheme of approximation and using (27) and (28) we can show that

$$\eta_x=-\frac{\mu}{r\left(1-\nu^2\right)},\ \eta_z=-\frac{\lambda}{r\left(1-\nu^2\right)},\ \eta_z=0 \tag{30}$$

and $$\zeta_x=\frac{\lambda\nu L}{r\left(1-\nu^2\right)^{\frac{1}{2}}},\ \zeta_y=\frac{\mu\nu L}{r\left(1-\nu^2\right)^{\frac{1}{2}}},\ \zeta_z=-\frac{M\left(1-\nu^2\right)^{\frac{1}{2}}}{r}, \tag{31}$$

where

$$L=1+\frac{r^3}{41580}\Big\{27720\sum_{i=1}^{3}a_i-33264\sum_{i=1}^{3}a_ib_ir^2\left(1-\nu^2\right)+17820\sum_{i=1}^{3}a_ib_i^2r^4\left(1-\nu^2\right)^2$$
$$-6160\sum_{i=1}^{3}a_ib_i^3r^6\left(1-\nu^2\right)^3+1575\sum_{i=1}^{3}a_ib_i^4r^8\left(1-\nu^2\right)^4-\ldots\Big\}$$

$$M=1-\frac{r^3}{83160}\Big\{27720\sum_{i=1}^{3}a_i-16632\sum_{i=1}^{3}a_ib_ir^2\left(1-\nu^2\right)+5940\sum_{i=1}^{3}a_ib_i^2r^4\left(1-\nu^2\right)^2$$
$$-1540\sum_{i=1}^{3}a_ib_i^3r^6\left(1-\nu^2\right)^3+315\sum_{i=1}^{3}a_ib_i^4r^8\left(1-\nu^2\right)^4-\ldots..\Big\}$$

Now from Eq. (7), we get

$$h_2=r_0\,sin\,\zeta\Bigg[1-r_0^3\Bigg\{\frac{1}{3}\sum_{i=1}^{3}a_i-\frac{1}{5}\sum_{i=1}^{3}a_ib_ir_0^2\,sin^2\zeta+\frac{1}{14}\sum_{i=1}^{3}a_ib_i^2r_0^4\,sin^4\zeta$$

$$-\frac{1}{54}\sum_{i=1}^{3}a_ib_i^3r_0^6\,sin^6\zeta\Bigg\}+r_0^3\,sin^2\zeta\Bigg\{\sum_{i=1}^{3}\left(\frac{5}{6}a_i\right)$$

$$-\sum_{i=1}^{3}\left(\frac{17}{732}a_ib_i^3\right)r_0^6\,sin^6\zeta\Bigg\}+r_0^4\Bigg\{-\frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jr_0^2\,sin^2\zeta$$

$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{11}{60}a_ia_jb_j\right)r_0^4\,sin^4\zeta-\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigg(\frac{4}{63}a_ia_jb_j^2$$

$$+\frac{1}{20}a_ib_ia_jb_j\Bigg)r_0^6\,sin^6\zeta\Bigg\}+r_0^4\,sin^2\zeta\Bigg\{-\frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jr_0^2\,sin^2\zeta$$

$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{11}{60}a_ia_jb_j\right)r_0^4\,sin^4\zeta-\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigg(\frac{4}{63}a_ia_jb_j^2$$

$$
\begin{aligned}
&+\frac{1}{20}a_i b_i a_j b_j \Bigg) r_0^6 \sin^6\zeta \Bigg\} - r_0^6 \cos^4\zeta \Bigg\{ \frac{1}{18}\Bigg(\sum_{i=1}^{3} a_i^2 + 2\sum_{i=1}^{3} a_i a_j \Bigg) \\
&-\frac{1}{15}\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j b_j r_0^2 \sin^2\zeta + \frac{1}{36}\Bigg(\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j^2 \\
&+2\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j a_k \Bigg) r_0^3 \sin^2\zeta + \frac{1}{50}\Bigg(\sum_{i=1}^{3} a_i^2 b_i^2 \\
&+2\sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j b_j \Bigg) r_0^4 \sin^4\zeta - \Bigg(\frac{1}{30}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i a_j a_k b_k \\
&+\frac{1}{72}\sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j^2 + \frac{1}{36}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i b_i a_j a_k \Bigg) r_0^5 \sin^4\zeta \Bigg\} \\
&-r_0^9 \cos^6\zeta \Bigg\{ \frac{1}{54}\Bigg(\sum_{i=1}^{3} a_i^3 + 3\sum_{i=1}^{3}\sum_{j=1}^{3} a_i^2 a_j + 6a_1 a_2 a_3 \Bigg) + \frac{1}{108}\Bigg(\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j^3 \\
&+3\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i a_k a_j^2 + 6a_1 a_2 a_3 \sum_{i=1}^{3} a_i \Bigg) r_0^3 \sin^2\zeta \Bigg\} - \frac{5}{648}\Bigg(\sum_{i=1}^{3} a_i^4 \\
&+18\sum_{i=1}^{3}\sum_{j=1}^{3} a_i^2 a_j^2 + 24\sum_{i=1}^{3}\sum_{j=1}^{3} a_i^3 a_j + 36\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3} a_i^2 a_j a_k \Bigg) r_0^{12} \cos^8\zeta + \ldots \Bigg]
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
h_3 = r_0 \Bigg[&1 - \Bigg(\frac{2}{3} - \sin^2\zeta \Bigg) \sum_{i=1}^{3} a_i r_0^3 + \Bigg(\frac{4}{5} - \sin^2\zeta \Bigg) \sum_{i=1}^{3} a_i b_i r_0^5 \sin^2\zeta + \Bigg(\frac{4}{9} - \frac{11}{6}\sin^2\zeta \\
&+\frac{3}{2}\sin^4\zeta \Bigg)\Bigg(\sum_{i=1}^{3} a_i^2 + 2\sum_{i=1}^{3}\sum_{j=i}^{3} a_i a_j \Bigg) r_0^6 - \Bigg(\frac{3}{7} - \frac{1}{2}\sin^2\zeta \Bigg) \sum_{i=1}^{3} a_i b_i^2 r_0^7 \sin^4\zeta \\
&-\frac{1}{6}\sin^2\zeta \Bigg) \sum_{i=1}^{3} a_i b_i^3 \sin^6\zeta + \Bigg(\frac{4}{9} - \sin^2\zeta + \frac{1}{2}\sin^4\zeta \Bigg)\Bigg(\sum_{i=1}^{3}\sum_{j=i}^{3} a_i a_j^2 \\
&+2\sum_{i=1}^{3}\sum_{j=i}^{3} a_i a_j a_k \Bigg) - \frac{5}{16}\Bigg(\frac{4}{5} - \sin^2\zeta \Bigg)^3 \Bigg(\sum_{i=1}^{3} a_i^3 + 3\sum_{i=1}^{3}\sum_{j=i}^{3} a_i^2 a_j \\
&+6a_1 a_2 a_3 \Bigg)\Bigg\} r_0^9 + \Bigg(\frac{16}{25} - \frac{21}{10}\sin^2\zeta + \frac{3}{2}\sin^4\zeta \Bigg)\Bigg(\sum_{i=1}^{3} a_i^2 b_i^2 \\
&+2\sum_{i=1}^{3}\sum_{j=i}^{3} a_i b_i a_j b_j \Bigg) r_0^{10} \sin^4\zeta - \Bigg(\frac{3}{5} - \frac{3}{4}\sin^2\zeta \Bigg)\Bigg(\frac{8}{9} \\
&-\frac{2}{3}\sin^2\zeta \Bigg)\Bigg(\sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_i a_j^2 + 2\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i b_i a_j a_k \Bigg) r_0^{11} \sin^2\zeta \\
&+\Bigg\{ \frac{3}{8}\Bigg(\frac{4}{9} - \frac{1}{3}\sin^2\zeta \Bigg)^2 \Bigg(\sum_{i=1}^{3} a_i^4 + 6\sum_{i=1}^{3}\sum_{j=i}^{3} a_i^2 a_j^2 + 12\sum_{i=1}^{3}\sum_{j,k=1}^{3} a_i^2 a_j a_k \Bigg)
\end{aligned}
$$

$$+\frac{35}{128}\left(\frac{4}{3}-2sin^2\zeta\right)^4\left(\sum_{i=1}^{3}a_i^4+18\sum_{i=1}^{3}\sum_{j=i}^{3}a_i^2a_j^2+24\sum_{i=1}^{3}\sum_{j,k=1}^{3}a_i^3a_j\right.$$

$$\left.\left.\left.+36\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}a_i^2a_ja_k\right)\right\}r_0^{12}+...\right] \tag{33}$$

Equations (29), (32) and (33) give explicit expressions for metric coefficients h_1, h_2 and h_3 associated with the system of Roche-coordinates (ξ, η, ζ) for the Roche-model of a star distorted by differential rotation.

5. Conclusion

The study revealed analytical expressions of ξ, η and, ζ as Roche-coordinates, and metric coefficients h_1, h_2 and h_3 associated with the double stars problems. The expressions for ξ and η are found to be exact, whereas the expression for ζ coordinate is correct up to second order terms.

This study also insists that the results may be applicable to the polytropic stars which are distorted by differential rotation such that the whole mass is supposed to be concentrated at the centre and this point mass is surrounded by an evanescent envelope.

It may be pointed out that although we have studied the problem of Roche coordinates associated with the equipotential surfaces by assuming the Roche model of the star, the present method of Roche coordinates can also be used when some more realistic structures is assumed for the interior of the problem. The paper has been concerned primarily with the geometry of the Roche-equipotential surfaces surrounding a gravitational dipole. This geometry may found many applications to the phenomenon exhibited by the close binary system.

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